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THREE-YEAR REPORT

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CENTER FOR ANALYSIS OF  
HETEROGENEOUS AND NONLINEAR MEDIA

Courant Institute of Mathematical Sciences  
New York University  
New York, NY 10012

Chief, Technical Information Division

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Courant Institute of Mathematical Sciences

New York University

New York, NY 10012

A University Research Initiative Center

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## Table of Contents

I.	Introduction .....	3
II.	Focus and Goals .....	4
III	Personnel	
	Faculty .....	5
	Postdoctoral Researchers .....	5
	Visitors .....	5
	Students .....	6
IV.	Synopses of Research Results .....	7
V.	Computational Environment .....	9
VI.	Workshops	
	Workshop on Mathematical Aspects of Vortex Dynamics .....	11
	Workshop on Random Media and Composites .....	13
VII.	Descriptions of Research Projects	
	Fluid Dynamics .....	15
	Numerical Methods for Fluid Dynamics .....	34
	Random Media and Composite Materials .....	44
	Nonlinear Optics .....	64
VIII.	Faculty Resumes .....	69
IX.	Publications of the Center .....	77



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## CENTER FOR ANALYSIS OF HETEROGENEOUS AND NONLINEAR MEDIA

### I. Introduction

The Center for Analysis of Heterogeneous and Nonlinear Media was established in October 1986 through the University Research Initiative (URI) Program. Funding for the Center is primarily from the Air Force Office of Scientific Research through the URI Program and the Applied Analysis Program managed by Arje Nachman.

The Center consists of a number of faculty, postdoctoral researchers, visiting scientists and graduate students integrated within the Courant Institute. Establishment of this Center provided a scientific focus and a unified computational environment of Sun workstations. The result has been a very active and collaborative research group. Numerical computation has played an extremely important role in the research of this Center, for simulation of physical systems and as an inspiration and guide for modeling and analysis.

The Courant Institute has provided a fruitful setting for the Center. Many faculty, postdoctoral visitors and graduate students of the Institute have become involved in the research projects of the Center. The Institute's extensive system of seminars and courses has helped keep the Center at the forefront of current scientific developments.

The scientific research of the Center has consisted of modeling, mathematical analysis and computation of problems in composite and random materials, nonlinear optics and fluid dynamics. The results of this research are presented in a number of research publications and several books. Two workshops, on vortex dynamics and on composite materials, were organized by the Center. These workshops provided a very useful forum for presentation of the results of the Center, as well as for scientific interchange among a wide range of participants.

## II. Focus and Goals

The focus of the Center has been on the following research areas:

- Vortex dynamics in incompressible fluids
- Numerical methods for fluid dynamics
- Macroscopic properties of composite materials
- Wave propagation in random media
- Nonlinear phenomena in optical materials.

Basic research in these subjects is of great technological importance and poses mathematical, numerical and physical challenges. In the research of the Center, we have maintained a balanced approach of physical modeling, mathematical analysis and numerical computation. The applications of these subjects have served as an important guide to the research problems. The interaction of analysis and computation has been one of the most successful features of the Center.

The research on these subjects has aimed toward the following goals:

- Development of mathematical models
- Mathematical analysis of physical systems
- Development and analysis of numerical methods
- Computation of particular solutions
- Computation of solutions when analysis is impossible.

These goals have been met in a number of research problems, which are described in Sections IV and VII. Two key elements in the success of the Center have been the strong computational environment that we have assembled and the high quality postdoctoral visitors and graduate students that we were able to attract.

Another important goal has been to generate interest in these research subjects among students and other mathematicians and scientists. This has been accomplished through numerous graduate courses, seminars and publications and two workshops.

### III. Personnel

The core of the Center consists of 4 faculty members:

Russel Caflisch

Graeme Milton

George Papanicolaou

Lu Ting

While a member of the Center, Graeme Milton was awarded the prestigious Packard Fellowship for Science and Engineering, as well as a Sloan Fellowship. Russel Caflisch received a Sloan Fellowship.

Postdoctoral researchers in the Center have included the following:

Marco Avellaneda (1987-88)

Thomas Chacon (1988)

Tom Hou (1987-89)

Robert Krasny (1987)

Michael Landman (1987-89)

John Lowengrub (1988-89)

Robert Palais (1986-87)

Demitri Papageorgiou (1986-87)

Bruce Pitman (1986-87)

Marie Postel (1988-89)

Sophie Weinryb (1988-89)

Marco Avellaneda joined the faculty of the Courant Institute as an assistant professor in Fall 1988. Tom Hou will be joining the faculty of the Courant Institute as an assistant professor in Fall 1989.

Long-term scientific visitors to the Center have included:

James Berryman (Lawrence Livermore National Laboratory)

Alexander Frenkel (City College, New York)

Christos Flytzanis (Ecole Polytechnique, France)

Brenton LeMesurier (University of Arizona)

Oscar Orellana (Santa Maria University, Chile)

Dale Pullin (University of Queensland, Australia)

Gregory Sivashinsky (City College, New York)

Pierre-Louis Sulem (University of Tel-Aviv, Israel)

Catherine Sulem (Ben-Gurion University, Israel)

Graduate students (and their Ph.D. date and research project) in the Center have included:

Mahmoud Affouf (Ph.D. 1988, computations of dynamic phase transitions)

Mark Asch (pulse reflection from random media)

Albert Fannjiang (nonlinear random media)

Robert Knapp (Ph.D. 1988, nonlinear random media)

John Lowengrub (Ph.D. 1988, vortex methods)

William Morokoff (numerical methods for the Boltzmann equation)

Andre Nachbin (Ph.D. 1989, gravity waves in shallow water with irregular bottom)

Vincenzo Nesi (Ph.D. 1989, optimal bounds for composites)

Oscar Orellana (Ph.D. 1987, vortex sheets)

Gabriella Puppo (Ph.D. 1989, computation of boundary layers)

Julia Rennenkampf (pulse reflection from random media)

Michael Siegel (Ph.D. 1989, singularities in Rayleigh-Taylor flow)

Xiao Wang (analysis and computation of singular solutions of the nonlinear Schroedinger equation)

Xue Xin ( nonlinear homogenization)

Jing-Yi Zhu (Ph.D. 1989, adaptive vortex method)

#### IV. Synopses of Research Results

Here we give synopses of several outstanding research results of the Center. More detailed descriptions of these and other research projects is found in section VII. Some of the most important results of the Center have been the following:

1. Computations and numerical analysis of the vortex method for vortex sheets were carried out by Krasny and by Caflisch and Lowengrub.
2. Exact singular solutions of the vortex sheet equations were found by Caflisch and Orellana.
3. Caflisch and Siegel have developed a theory for the formation of singularities in Rayleigh-Taylor flows, which gives a prediction of their form and time of development.
4. The stability analysis of the classical three point-vortex problem was completed by Tavantzis and Ting.
5. Ting and coworkers have analyzed reflection and transmission of an acoustic wave through a bubbly layer in a liquid.
6. Convergence of the point vortex method for inviscid, incompressible fluid flow was proved by Goodman, Hou and Lowengrub. This new result is expected to greatly change the understanding and use of vortex methods.
7. An adaptive vortex method was developed, implemented and analyzed by Hou, Papageorgiou, Papanicolaou and Zhu.
8. Papanicolaou and Zhu have developed and implemented a particle method for computation of dilute fluid-particle flows.
9. A finite element vortex method was developed, implemented and analyzed by Chacon and Hou.
10. The formulation and solution of inverse problems for pulse reflection from random media was developed by Papanicolaou and coworkers.
11. Nachbin and Papanicolaou have developed and implemented a computational method for water waves in a channel with an irregular bottom.
12. Kohn and Milton have found generalizations of the Hashin-Shtrikman bounds for anisotropic materials.
13. Optimal upper and lower bounds on the conductivity of a polycrystalline material were derived by Milton and coworkers.

14. Milton has derived a macroscopic theory of the Hall effect in composite materials.

15. A numerical homogenization method originally developed by Engquist and Hou for hyperbolic equations was extended by Avellaneda, Hou and Papanicolaou to elliptic equations in several dimensions.

16. The form of the focusing singularity of the nonlinear Schroedinger equation was determined by Landman, Le Mesurier, Papanicolaou, Sulem and Sulem.

17. Localization and bistability for a nonlinear, random optical media was analyzed and simulated by Knapp, Papanicolaou and White.

## V. Computational Environment

The computational environment has been a very important component of the Center. With funds from the AFOSR and the NSF and with substantial contributions from New York University, the Center acquired a system of SUN 3 workstations linked by an ethernet, a Celerity mini computer, an Iris graphics workstation and an array of peripherals and software.

Ease of use of the workstations and general availability of computational cycles has transformed the role of computations in our research efforts. In addition to their traditional use in providing highly accurate solution of well-formulated systems, computations now serve as an experimental mathematical tool for investigating phenomena that are poorly understood and for testing hypothetical properties. This experimental role for computations has been particularly strong, for example, in the our investigations of singularity formation for the nonlinear Schroedinger equation, singularity formation for vortex sheets, and the solution of inverse problems in random media.

The principal hardware components of our computer system are

1 Sun-3/180 fileserver and color workstation (the main server at present)

10 Sun-3/50 and 1 SUN-3/60 (purchased with NSF funds) workstations

1 SUN-3/280 server (purchased with CIMS funds but not fully equipped at present)

1 Celerity 1200 mini computer (purchased with NSF funds in 1985)

1 Iris 4D/50-GT graphics workstation.

In addition to the standard UNIX-based software distributed with the machines, we have several public-domain software packages (cmplib, auto, near graphics, etc.) and the following software items that we purchased:

Precision Visuals DI3000, PICSURE and Metafile translator (graphics package)

NAG library of scientific routines.

Graphics animation system (GAS) and Surface plot system (SURF) for the IRIS, from Sterling Software

Our supercomputing is done at the John von Neumann Center at Princeton and is sponsored by the NSF. We have a 1.5 megabit/second communications link to Princeton's Cyber 205's and ETA 10's. The speed of data transmission to and from Princeton make it the preferred supercomputing center for us. We have accounts at San Diego, NASA-Ames, Kirtland AFB and Pittsburgh, but the links are slow and so we

do not use them very much, even though they have Cray's running UNIX making them much more convenient for us.

## VI. Workshops

Two workshops were organized by the Center:

### Workshop on Mathematical Aspects of Vortex Dynamics

The Workshop on Mathematical Aspects of Vortex Dynamics was held at the Xerox Training Center in Leesburg, Virginia, on April 25 - 27, 1988. The Center and the Society for Industrial and Applied Mathematics (SIAM) were joint sponsors. The workshop brought together researchers in mathematical analysis, scientific computing and theoretical physics and engineering, from universities, industry and government research laboratories, who are working on vortex dynamics in fluids. A large variety of fluid problems were discussed, such as vortex sheets and shear layers, the vortex reconnection problem, vortex methods, coherent vortex structures, fluid dynamic stability, systems of point vortices, turbulence modeling and other applications. The results presented included mathematical existence theorems, results on convergence of numerical methods, computations and qualitative analysis of flows.

Topics and speakers for the meeting were selected by a Program Committee consisting of Hassan Aref (UC San Diego), Gregory Baker (Exxon Corporate Research and Ohio State University), J. Thomas Beale (Duke University), Russel Caflisch (chairman, Courant Institute), Steven Childress (Courant Institute), and Sidney Leibovich (Cornell University).

There were approximately 60 attendees at the workshop, representing universities, government and industry. The speakers and the titles of their presentations were the following:

1. C. Anderson, (University of California, Los Angeles), "An Investigation of the Accuracy of Vorticity Boundary Condition"
2. B. J. Bayly, (Courant Institute), "Computations of Broad-Band Instabilities in a Class of Closed-Streamline Flows"
3. M. Berger (U. Massachusetts), "Remarks on Vortex Breakdown"
4. T. Butke, (Princeton University), "Turbulence in Super Fluids"
5. L. J. Campbell, (Los Alamos National Laboratory), "Vortex Lattices in Theory and Practice"
6. G.H. Cottet, (UCLA and University of Paris), "New Boundary Conditions for Navier Stokes Equations in the Vorticity Formulation"
7. D. Dritschel, (Cambridge University), "Stripping of Two-Dimensional Vortices by Weak Strain and Shear"
8. T. Kambe, (University of Tokyo), "Analysis of Vortex-Sheet Motions Based on the Theory of Hyperfunctions"
9. A. Leonard and G. Winckelmans, (California Institute of Technology), "Improved Vortex Methods for Three-Dimensional Flows with Application to the Interactions of Two Vortex Rings"
10. D.G. Long, (Princeton University), "The Random Vortex Method in Two Dimensions"

11. J. Lowengrub, (Courant Institute), "Convergence of the Vortex Method for Vortex Sheets"
12. J. L. Lumley, (Cornell University), "The Dynamics of Vortex Structures in the Wall Region of a Turbulent Boundary Layer"
13. T. S. Lundgren, (University of Minnesota), "A Free Surface Vortex Method With Small Viscous Effects"
14. P. S. Marcus, (University of California, Berkeley), "Vortex Dynamics in Shearing Flows"
15. D. I. Meiron, (California Institute of Technology), S. A. Orszag, M. J. Shelley, (Princeton University), "A Numerical Study of Vortex Reconnection"
16. Richard Peltz, (Rutgers University), "Spectral Methods on Hypercube Computers"
17. C. Pozrikidis, (University of California at San Diego), "Two-Dimensional Vortex Dynamics in Flows of Mixed Shear Layer-Wake Type"
18. D. I. Pullin, (The University of Queensland, Australia), "On Similarity Solutions for the Self-Induced Motion of Two-Branched Vortex Sheets"
19. M. Pulvirenti, (Universita dell'Aquila, Italy), "On the Invariant Measures for the 2-D Euler Flow"
20. M. J. Shelley and C. R. Baker, (Princeton University), "On the Connection between Thin Vortex Layers and Vortex Sheets"
21. G. Tryggvason, (The University of Michigan), "Vortex Dynamics of Stratified Flows"
22. B. Turkington, A. Eydeland, (University of Massachusetts), "An Iterative Method For Computing Steady Vortex Flow Systems"

The proceedings of the workshop will be published in the SIAM Conference Proceedings Series.

## Workshop on Random Media and Composites

Composite and random media include a wide variety of materials ranging from alloys, emulsions and colloidal suspensions to polycrystalline aggregates, porous rocks and fibrous substances. The Air Force URI Center at the Courant Institute and the Society for Industrial and Applied Mathematics (SIAM) held a conference on random media and composites from December 7th to 10th 1988 inclusive, at the Xerox Training Center in Leesburg, Virginia. The main emphasis of the conference was on theoretical developments in understanding the effective moduli of composites and on localization in random media. The scope also extended to percolation, to non-linear effects and micromechanics and to composite effects in superconductivity.

The organizing committee for the scientific program of the conference consisted of Graeme Milton (Courant Institute), chairman; David Johnson (Schlumberger-Doll Research Laboratories); Robert Kohn (Courant Institute); George Papanicolaou (Courant Institute); Pabitra Sen (Schlumberger-Doll Research Laboratories); Ping Sheng (Exxon Corporate Research); Luc Tartar (Carnegie-Mellon University); Benjamin White (Exxon Corporate Research). The committee represented a balance of research scientists from universities and industry. Johnson, Milton, Sen and Sheng have close connections with the physics community while Kohn, Papanicolaou, Tartar, and White are most familiar with recent mathematical developments.

There were approximately 80 attendees at the workshop, representing universities, government and industry. The speakers and the titles of their presentations were the following:

1. A. Acrivos, (City University of New York, New York), "An Effective Continuum Theory for Estimating Transport Parameters in Two-Phase Random Media"
2. M. Avellaneda, (Courant Institute), "Optimal Bounds and Microgeometries for Polycrystalline Aggregates"
3. R. Barrera, (Universidad Nacional de Mexico), "An Intuitive Approach and a Diagrammatic Formulation of the Renormalized Polarizability in the Maxwell-Garnett Theory"
4. D. Bergman, (Tel-Aviv University, Israel), "Nonlinear Behavior and 1/f Noise Near a Conductivity Threshold--The Importance of Local Geometry and the Failure of the Effective Medium Theory"
5. P.M. Duxbury, (Michigan State University), "Scaling Theory and Fluctuations in the Strength of Composites"
6. D. Fisher, (Princeton University), "Onset of Superfluidity in Random Media"
7. K. Golden, (Princeton University), "Convexity in Random Resistor Networks"
8. L. Greene, (Bell Communications Research), "High Temperature Superconductivity"
9. B. Halperin, (Harvard University), "Percolation and Transport in Highly Disordered Systems"
10. R. James, (University of Minnesota, Minneapolis), "Theory of Martensitic Transformations"
11. A. Lagendijk, (University of Amsterdam, The Netherlands), "Experiments on the Localization of Light"

12. C. Lobb, (Harvard University), "Simulations of Composites: From Ohmic Response to the Break-down of Superconductivity"
13. J. Machta, (University of Massachusetts), "Localization of the Sound Modes of Helium in Porous Media and Symmetry Breaking by a Flow"
14. J. Maynard, (Pennsylvania State University), "Acoustic Anderson Localization"
15. W. Means, (State University of New York), "Synkinematic Microscopy of Transparent Polycrystals"
16. F. Murat, (University of Paris VI, France), "Homogenization in Thermoelasticity"
17. T. W. Noh, (Cornell University), "Infrared Electrodynamical Properties of High  $T_c$  Superconductors"
18. L. Poladian, (University of Sydney, Australia), "Critical Behaviour of Effective Moduli Determined from Nearest Neighbour Interactions"
19. S. Redner, (Boston University), "Transport and Dispersion in Random Media"
20. J. Rubinstein, (Israel Institute of Technology), "Bounds on Various Electrostatic and Hydrodynamic Capacities"
21. D. Stroud, (Ohio State University), "Theory of the Transport and A.C. Properties of Superconducting Composites"
22. W. Symes, (Rice University), "Propagation of Regularity and Inverse Problems"
23. D. Tanner, (University of Florida, Gainesville), "Infrared Studies of High- $T_c$  Materials"
24. B. White, (Exxon Research and Engineering Co.), "Pulse Reflection from a Randomly-Stratified Medium"
25. John Willis, (University of Bath, England), "The Overall Behaviour of a Nonlinear Matrix Reinforced by Rigid Inclusions or Weakened by Cavities"
26. P.Z. Wong, (University of Massachusetts), "Transport Properties of Real and Artificial Rocks"

The proceedings of the workshop will be published in the SIAM Conference Proceedings Series.

## VII. Descriptions of Research Projects

### VII.1 Fluid Dynamics

#### (1) Analysis and Computation for Vortex Sheets (Caflisch, Krasny, Lowengrub, Orellana)

A vortex sheet in two-dimensional, inviscid, incompressible fluid flow is a curve along which the velocity is discontinuous in its tangential component. Vortex sheets are generated in a variety of laboratory and natural flows, such as mixing flow past a splitter plate, separated boundary layers, and flow over delta wings. The instability of vortex sheets is one of the primary mechanisms for generation of small scales and the onset of turbulence in high Reynolds number flows. In fact the growth rate for this instability can be arbitrarily large since it is proportional to the wavenumber.

This research project is on computation and analysis of this strongly unstable (even ill-posed) problem. In particular the vortex sheet develops a singularity (a point at which the curvature of the sheet is infinite) and then rolls-up into a spiral. The main research goals are the following:

- (1) Determination of a mathematical formulation in which the vortex sheet problem is well-posed.
- (2) Development of numerical methods for the vortex sheet problem, analysis of convergence of these methods, and investigation of particular physical phenomena by numerical computation.
- (3) Construction of examples of singularity formation on vortex sheets.
- (4) Determination of the generic form for singularity formation.
- (5) Investigation of the behavior of the vortex sheet after the singularity formation, including roll-up.

As described below, (1-3) are fairly well completed; while we have only partial results on (4) and (5). This project has been initiated by the numerical investigations of Krasny[8,9]. All of the analysis has been inspired by his computational results.

One of the main mathematical motivations for this work is the importance of singularities for a variety of fluid problems. Most important is the possible development of singularities in three-dimensional inviscid flow, which would be related to the development of turbulence. On the other hand, in two dimensions singularities do not develop from smooth initial data. The vortex sheet problem is of intermediate difficulty: singular initial data (a vortex sheet) in two dimensions are found to become more singular after some time. An even stronger singularity, involving energy concentration, has been

proposed for two dimensional flows by DiPerna and Majda [4-6].

The vortex sheet is described by a complex function  $z(\gamma, t) = x(\gamma, t) + iy(\gamma, t)$  in which  $\gamma$  is the circulation variable and  $z$  is the complex position of the sheet. The evolution of the sheet is governed by the Birkhoff-Rott equation

$$\partial_t \overline{z(\gamma, t)} = B[z] = (2\pi i)^{-1} PV \int_{-\infty}^{\infty} (z(\gamma, t) - z(\gamma', t))^{-1} d\gamma' \quad (1)$$

in which the integral is a Cauchy principal value integral. An exact steady solution of this equation is  $z = \gamma$  corresponding to a flat vortex sheet of uniform strength. For a nearly flat vortex sheet the solution can be written as  $z = \gamma + s(\gamma, t)$  with  $s$  small. The principal linear modes for  $s$  are  $s_{\pm k} = (1 \pm i) \exp(ik\gamma \pm kt/2)$ . This Kelvin-Helmholtz instability has a linear growth rate that is proportional to the wavenumber  $k$  and hence can be arbitrarily large.

### Numerical Computations of Vortex Sheets

Krasny [8,9] developed two methods for computing the evolution of vortex sheets. The main numerical difficulty of this problem is that the physical instability of the vortex sheet can magnify the roundoff errors of the computation.

In the first method the roundoff errors are controlled by spectral filtering at a fixed filter level  $e_f$ . At each step of the computation the fourier component  $\hat{s}(k)$  is set to zero if its magnitude is below  $e_f$ . In this way the high wavenumber components grow only if the nonlinearity of the problem generates them. This method works up to the time of singularity formation and gives predictions on the time and type of singularity formation. Meiron, Baker and Orszag [10] also computed the form and time of the singularity by an alternative method.

Krasny's second method is a desingularization of the Birkhoff-Rott equation (1). Replace the integral operator  $B[z]$  in (1) by

$$B_\delta[z](\gamma) = \frac{1}{2\pi i} PV \int_{-\infty}^{\infty} \frac{\overline{z(\gamma) - z(\gamma')}}{|z(\gamma) - z(\gamma')|^2 + \delta^2} d\gamma' \quad (2)$$

which equals  $B[z]$  for  $\delta = 0$  and has no singularities for  $\delta > 0$ . Using this equation Krasny was able to compute past the singularity time and find rolled-up spiral solutions, as well as interactions of these

spirals.

### Well-Posedness of Vortex Sheets for Analytic Initial Data

Kelvin-Helmholtz instability makes the vortex sheet problem ill-posed in a Sobolev function space, but the problem is well-posed in the space of analytic functions. The reason for this is that the linearized Birkhoff-Rott equation is hyperbolic in the imaginary  $\gamma$  direction. Thus details of the solution move on characteristics at speed  $\pm i/2$ . Using this description of the problem, Caffisch and Orellana [1] proved the following long time existence theorem:

Suppose the initial data  $s(t=0, \gamma)$  is analytic and small in a strip  $\{|Im\gamma| < \rho\}$ . Then the Birkhoff-Rott equation (1) has an analytic solution  $z = \gamma + s(t, \gamma)$  for a time interval  $0 < t < (2-\alpha)\rho$ , in which  $\alpha$  is a small nonzero number.

This result is nearly optimal, since it is valid almost up to the observed singularity time. An earlier short-time version of this result for somewhat more general initial data was proved by Sulem, Sulem, Bardos and Frisch[13]. The proof is based on the abstract Cauchy-Kowalewski theorem.

Next consider Krasny's desingularized equation with time discretization size  $\Delta t$  and with simulated roundoff error  $e_r$ , given by

$$\tilde{z}((n+1)\Delta t) = \tilde{z}(n\Delta t) + \Delta t B_{\delta\Delta x}[\tilde{z}(n\Delta t)] + e_r, \quad (3)$$

in which  $B_{\delta\Delta x}$  is a discrete sum approximation to the integral  $B_\delta$ . Using the same methods, Caffisch and Lowengrub[3] proved the following convergence theorem for Krasny's numerical method:

Suppose as above that the initial data  $s(t=0, \gamma)$  is analytic and small in a strip  $\{|Im\gamma| < \rho\}$ . Also assume that the roundoff error satisfies

$$|e_r| < \Delta t e^{-1/\Delta x}. \quad (4)$$

Then  $|z - \tilde{z}| < c(\Delta t + \Delta x + \delta + \epsilon)$ .

This theorem is only for a short time, much shorter than the singularity time. The restriction (4) on the roundoff error size is the numerical analysis interpretation of the restriction to analytic functions.

## Singularities - Examples and the Generic Form of Singularities

Singular solutions of the Birkhoff-Rott equation (1) were constructed by Caffisch and Orellana [2] and by Duchon and Robert [7]. These solutions are analytic in  $\gamma$  for  $t = 0$ , and at the critical time  $t = t_c$  they have the form  $z = \gamma + \varepsilon \operatorname{sgn}(\gamma) |\gamma|^\alpha$  for  $\gamma \approx 0$  in which  $\varepsilon$  is small. Such solutions can be produced for any  $\alpha > 1$ ; i.e. singularities can be produced in anything stronger than the first derivative. In particular the curvature of the sheet will be infinite at the singularity.

These solutions are not believed to be typical. An asymptotic analysis by Moore [11,12] suggests that the typical form of singularities has  $\alpha = 3/2$ . The reason is that the singularity occurs as an envelope of characteristics, and on such an envelope the typical behavior is square root type. A precise mathematical analysis of this question is currently being carried out by Caffisch, Ercolani, Hou and Semmes. Their analysis is based on an approximation of the Birkhoff-Rott equation by a system of nonlinear hyperbolic (in the imaginary  $\gamma$  direction) equations of the form

$$\partial_t F = M(F) \partial_\gamma F. \quad (5)$$

The vector  $F$  is  $(s, s^*, Hs, Hs^*)$  in which  $s(\gamma) = \overline{s(\bar{\gamma})}$  and  $Hs$  is the Hilbert transform of  $s$ .

The vortex sheet singularity and subsequent roll-up of the sheet act as a strong source of noise in a slightly compressible fluid. An analysis of the resulting sound waves is found in [14].

Caffisch is currently writing a monograph on the mathematical and numerical aspects of vortex dynamics.

## References

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## (2) Singularity Formation for Rayleigh-Taylor Flows (Caflish, Siegel)

The Rayleigh-Taylor problem consists of two fluids of different densities  $\rho_1$  and  $\rho_2$  separated by an interface and acted on by gravity  $g$  or some other external force. This is a fundamental problem of fluid dynamics that arises in a large variety of applications, such as a blast wave propagating through an interface.

A steady solution of the problem is for the interface to be flat and stationary with a hydrostatic pressure distribution. However this solution is unstable, if the top density  $\rho_1$  is larger than the bottom density  $\rho_2$ . As a result fingers of the heavier top fluid are pushed into the lighter bottom fluid. The resulting type of fingering is strongly dependent on the value of the Atwood number  $A$ , defined by  $A = (\rho_1 - \rho_2) / \rho_1$

As the finger pushes into the bottom fluid, vorticity is baroclinically generated on the interface so that the interface is unstable due to Kelvin-Helmholtz instability. Thus if surface tension and viscosity are ignored, we expect singularities to form along the interface, followed by roll-up of the interface, as described in subsection (1). Vorticity is of opposite sign on the two sides of a finger, so that singularities of opposite sign are expected to form on the two sides. This process is important as a mechanism for generation of small scale structures in the flow. If surface tension or viscosity were small but nonzero, the singularity would be slightly smoothed-out, but the generation of small scales would still be present.

This roll-up has been experimentally observed and numerically computed [1]. Singularities are more difficult to compute. But in the Boussinesq approximation with  $A \rightarrow 0$ ,  $Ag \rightarrow 1$ , singularities have been computed by Pugh [5], a student of Derek Moore. At the other extreme,  $A = 1$ , it is observed that the positions of the singularities on the two sides of the finger tend to merge at the tip. Since the two singularities come with opposite sign, it has been conjectured [1] that they will cancel and that the flow for  $A = 1$  will be smooth.

In his Ph.D. thesis under the direction of Caflish, Siegel [6] has developed a simple approximate theory to describe the Rayleigh-Taylor problem. This theory is a particular case of a more general approximation method developed in [2]. Whereas the Rayleigh-Taylor equations are nonlocal integro-differential equations, the approximate theory consists of a system of first order pde's. For a nearly flat interface these pde's can be approximately solved by perturbation theory. A singularity is seen to form as an envelope of the characteristics of the pde, analogous to the singularity for the Kelvin-Helmholtz equation described by Moore [3,4]. In this way Siegel finds an expression for the time and type of singularity formation, the first such analytic expression for this problem. This result completely agrees with Pugh's

numerical results for the Boussinesq limit.

Siegel is currently analyzing the cancellation of the singularity for  $A = 1$  and carrying out more extensive computations to verify his analytic results. It may also be possible to include surface tension effects.

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### (3) Dynamics of Point Vortices (Ting)

The motion of  $N$  free point vortices in a two dimensional incompressible inviscid fluid and their induced flow field has been a subject of interest in studies of Hamiltonian dynamical systems, numerical simulations of chaotic motions and physical applications (see e.g., the recent review paper by Aref [1] and the references therein). The dynamics of three point vortices was studied by Groebli [2] (1877) and by Synge [3] (1949). Unaware of the earlier works, Novikov [4] (1975) and Aref [5] (1979) rediscovered the results obtained by Groebli and Synge. In Synge's analysis [3], the lengths of the sides of the triangle formed by the three vortices were used as prime variables. The critical states at which the lengths of the sides remain fixed throughout the motion were found to be either equilateral triangles or collinear configurations. The equilateral configurations were shown to be stable or unstable depending on whether the sum of the products of strengths  $K$  was greater or less than zero respectively. In the case  $K = 0$ , a one parameter family of solutions of contracting and another of expanding similar triangles were found.

Ting and Tavantzis [6] continued Synge's analysis so as to obtain the global behavior of the trajectories representing the configurations formed by the three vortices. They are able to completely classify the trajectories given the strengths of the three vortices and describe the trajectory qualitatively once the initial configuration is specified. They show that for the special case of  $K = 0$ , the family of contracting similar solutions is always unstable while the family of expanding ones is stable. They then study the critical states for collinear configurations in the general case where  $K$  is greater or less than 0, and show that there are either six or four critical states depending on the strengths of the vortices. When there are six collinear critical states, three of them are always stable, one is not, while the remaining two are unstable (stable) if the equilateral triangle configuration is stable (unstable). When there are only four collinear critical states, they all are stable while the equilateral triangle configuration is always unstable. Since there are two equilateral triangle configurations, clockwise and counter-clockwise arrangements of the three vortices, the sum of the indices of all the critical states is equal to +2 regardless of whether  $K$  is greater or less than 0. An integral invariant in trilinear coordinates is derived. Knowing all the critical points and the integral invariant, we obtain the global behavior of the trajectories.

References are listed at the end of subsection (6).

#### (4) Motion and Interaction of Vortices with Diffusive Core Structures (Ting)

In April 25-29, 1988 ten invited lectures were given by L. Ting at the Aerodynamisches Institut, Rheinisch-Westfaelische Technische Hochschule Aachen in commemoration of the 75th anniversary of the founding of the Institute. The lecture series, entitled "*Motion and Interaction of Vortices with Diffusive Core Structures*", is based on the past research of Ting and his coworkers sponsored by AFOSR, ONR and NASA since 1964 (see references 7-17). The outlines of the ten lectures are :

- (1) The length and time scales of a vortex dominated flow, governing equations, the integral invariants, the far field behavior and the approximate boundary conditions for a computational domain [7].  
(Lecture I, II)
- (2) The classical inviscid theory, motion of a spinning disc, the Biot-Savart formula and the singular terms; diffusion of an isolated vortex and the optimum similarity solution [8]. (Lecture III)
- (3) Matched asymptotic solutions of two dimensional vortical flows, multiple length and time scales, velocity of a vortex center and its diffusive core structures, the optimum similarity solutions; the meaning of the leading and higher order solutions [9]. (Lecture IV, V)
- (4) Motion of slender vortex filaments, the coupling between the axial and circumferential velocity components in the core structure, the physical meaning, the matched asymptotic solutions, interaction of vortex filaments [10-12]. (Lecture VI, VII)
- (5) Practical limits of the asymptotic solutions, studies of the merging of vortices, the rules of merging [13-15]. (Lecture VIII)
- (6) Approximate solutions of N-S equations using vortices with diffusive core structures [16, 17].  
(Lecture IX)
- (7) Concluding remarks, discussions. (Lecture X)

An extended version of these lectures is being prepared for publication as a monograph in collaboration with Rupert Klein, who was at Aachen and is now a visiting scholar in the Department of Mathematics, Princeton University.

A minisymposium in the First International Congress of Applied Mathematics, 1987, was organized and chaired by Ting to discuss the full complexity of three dimensional vortical flows around aerodynamic shapes.

References are listed at the end of subsection (6).

### (5) Noise Generation from Vortical Flows (Ting)

A primary acoustic source in unsteady flows is the time-dependence of the vorticity. Analysis of this source of noise generation is of importance in a wide variety of applications, such as airplane noise.

Far field noise induced by unsteady vortical flows has been studied by many investigators interested in sound generation by turbulence. In his pioneering work, Lighthill [18] found that the leading order acoustic field should be quadrupoles. Later Ribner [19] pointed out that there could be an acoustic source due to fluid dilatation. An asymptotic analysis has been done by Crow [20] where he matched the inner low Mach number inviscid flow to an outer acoustic field and showed that for an inviscid isentropic flow, the leading order acoustic field could be composed of quadrupoles only.

In the investigation of Miksis and Ting [21] we study the acoustic field induced by an unsteady vortical flow with viscosity. We consider the limit where the reference length scale of the vortical flow is much smaller than the reference length scale of the acoustic field. At low Mach number,  $M$ , the leading order solution of the vortical flow is an incompressible viscous flow. We match the far field behavior of the flow [7] to the quadrupoles of the acoustic field and relate their strengths to linear combinations of second moments of vorticity. The compressibility effect of the vortical flow, of  $O(M^2)$ , induces an acoustic source which is of the same order as the quadrupoles. We also relate the strength of the acoustic source to the total dissipation energy of the incompressible vortical flow.

The leading order acoustic pressure is  $O(M^3)$  and is composed of acoustic quadrupoles and an acoustic source, i.e.

$$\frac{p - P_0}{\rho_0 C^2} \approx M^3 \tilde{p}^{(0)} = \tilde{\Phi}_i + \tilde{\phi}_i \quad (1)$$

$$= \frac{M^3 B_i(\theta, \psi, \tau)}{\tilde{r}^3} - \frac{M^3 \bar{m}'(\tau)}{4\pi \tilde{r}}$$

where  $\tau$  is the retarded time  $t - \tilde{r}$ . The first term represents the quadrupoles with strength

$$B(\theta, \psi, \tau) = \frac{1}{24\pi} \sum_{i=1}^3 (9F_i(\tau) \hat{x}_j \hat{x}_k + 2\hat{x}_i^2 (H_j(\tau) - H_k(\tau))),$$

with  $i, j, k$  in cyclic order. Here  $\hat{x}_i$ ,  $i=1,2,3$  denote the three components of  $\mathbf{x}/r$  and are defined by the spherical coordinates  $\theta$  and  $\psi$ . The strengths  $F_i$ ,  $H_i$  are related to the second moments of vorticity in the

inner region, i.e.

$$F_i(t) = \langle \omega_i(x_j^2 - x_k^2) \rangle \quad \text{and} \quad H_i(t) = \langle 2\omega_i x_j x_k - \omega_j x_k x_i - \omega_k x_i x_j \rangle$$

for  $i=1,2,3$  with  $i,j,k$  in cyclic order. We note that  $H_1 + H_2 + H_3 = 0$ , so that there are only five quadrupoles.

The second term in the acoustic pressure (1) represents an acoustic source with strength  $\bar{m}(\tau)$  related to the total dissipation energy of the inner region, i.e.

$$\bar{m} = \frac{\gamma-3}{3\gamma} \langle \Theta \rangle,$$

where

$$\Theta = \frac{1}{2\text{Re}} \sum_{j,k=1}^3 \left( \frac{\partial v_j}{\partial x_k} + \frac{\partial v_k}{\partial x_j} \right)^2$$

and

$$\langle \cdot \rangle = \iiint_{-\infty}^{\infty} \cdot \, dx_1 \, dx_2 \, dx_3 .$$

These results are applied to turbulent flows and their implications for the mean flow are discussed.

It is known that the far field behavior of the vector velocity potential induced by an exponentially decaying vorticity distribution is given by a power series in  $r^{-1}$  where  $r$  is the distance from the origin [7]. Now we can show that the  $n$ th term in the power series for the far field velocity depends only on  $2n+1$  linear combinations of the  $n$ th moments of vorticity, for which there are  $(3/2)(n+2)(n+1)$  of them. This investigation is being prepared for publication by R. Klein and L. Ting [22].

References are listed at the end of subsection (6).

## (6) Wave Propagation and Multiphase Flow (Ting)

In continuing our past research on wave propagation through bubbly media, we investigated the transmission and reflection of an acoustic wave through a bubbly layer. We use the nonlinear model equations of Van Wijngaarden, which were derived systematically by Caflisch, Miksis, Papanicolaou and Ting [23]. We obtain numerical solutions of the nonlinear system and compare them with the linearized solutions. We find that even for a small amplitude incident pressure wave, it is possible to have nonlinear transmitted and reflected waves. We also consider the limit where the bubbly layer is thin relative to the incident wave length  $\lambda$ .

By using matched asymptotic expansions we find that we can replace the bubbly layer by an interface subjected to the condition of continuity of pressure and a nonlinear jump condition, i.e.

$$p^+ = p^-$$

and

$$(\mathbf{u}^+ - \mathbf{u}^-) \cdot \mathbf{n} = \int_{-\infty}^{\infty} K \bar{R}^2 \bar{R}_t \, d\bar{n}$$

in which  $\mathbf{u}^\pm$  denote the velocities on the  $\pm$  sides of the thin bubbly layer,  $\mathbf{n}$  is the unit normal vector, and  $\bar{n}$  and  $\bar{R}$  are the distance normal to the layer and the bubble radius scaled by the reference length  $h$ . Here the parameter  $K$  is

$$K = \frac{3\beta_0 \rho c^2}{p_e} \frac{h}{\lambda}$$

where  $\beta_0$  is the reference gas volume fraction,  $\rho$  and  $c$  are the density and the speed of sound on the + side, and  $p_e$  is the equilibrium pressure. The equation for the jump of the normal velocity contains internal effects of the bubbly layer. Solutions of this limiting case are compared with the numerical results and good agreement is found even when the ratio of the bubbly layer thickness to the wave length is of order one. These results were presented at "A Workshop on Theoretical Aspects of Multiphase Flow Phenomena", at Cornell University, Mathematical Sciences Institute, in October 1988 and reported in [24].

We also derived an analytical model based on the principle of minimum energy to determine the steady-state two-phase mixture level and axial gas volume fraction  $\alpha$ . The gas is introduced uniformly

across the base of a shallow liquid pool and escapes through the upper interface while the mass of the liquid is conserved. The resulting Euler equation and the natural boundary condition at the free interface are reduced to algebraic equations, i.e.

$$-\frac{\zeta}{2\alpha^2} + \frac{3}{We} - \left(\frac{x}{Fr} + \lambda\right) = 0$$

$$\frac{\zeta}{2\alpha_1} + \frac{3\alpha_1}{We} + (1 - \alpha_1)\left(\frac{x_1}{Fr} + \lambda\right) = 0,$$

in which  $\zeta$  is the added mass coefficient. It was found that the mixture level  $x_1$  and the gas volume fraction  $\alpha(x)$  are functions of the Froude number  $Fr$  and the Weber number  $We$ , whereas the gas volume fraction  $\alpha_1$  just below the free interface depends only on the Weber number, i.e.

$$\alpha(x) = \left(\frac{\zeta/2}{3/We - \lambda - x/Fr}\right)^{1/2}$$

$$\alpha_1 = \frac{\zeta We}{6} \left\{ \left(1 + \frac{6}{\zeta We}\right)^{1/2} - 1 \right\}.$$

The gas volume fraction has an upper bound of 1/2 for the limiting case of a bubbly homogeneous flow for which the added mass coefficient  $\zeta$  approaches infinity. This agrees well with experimental results where bubbly flows undergo a phase change at an approximate void fraction of 0.45. This investigation [25] will be presented at the 26th National Heat Transfer Conference, August 6-9, 1989, Philadelphia.

During the period, September 1987 to May 1988, Ting was on sabbatical leave from NYU. While visiting Stanford University, he and J.B. Keller completed an investigation of slender jets and thin sheets with surface tension [26]. While visiting at Northwestern University, he and M.J. Miksis completed a study of scattering of an incident wave from an interface separating two fluids [27].

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### (7) Helical Fluid Flows (Landman)

An important class of fluid motions are those which are constrained to be helical, i.e., in standard cylindrical polar coordinates, the velocity of the fluid is of the form  $\mathbf{v}=\mathbf{v}(r, \theta+\alpha z, t)$ . By assuming this symmetry, the equations of motion can be reduced to an essentially 2-dimensional polar-coordinate representation; however the dynamics still allows the stretching of vortex lines, a 3-dimensional phenomenon which is of particular interest in the study of vortex tube interactions and turbulence.

A theoretical aspect of this work concerns the investigation of steady solutions of the inviscid Euler equations. In collaboration with Steve Childress and Hank Strauss [2], Landman has found that the form these equations take allows for the specification of two arbitrary functions to produce a steady helical vortex. The determination of these functions is possible if one considers the vortex as being formed in the limit of vanishing viscosity, known as the Prandtl-Batchelor limit. Prandtl [6] and Batchelor [1] analyzed the analogous problem for steady flow in 2-dimensions and with axisymmetry (both are limits of helical solutions as  $\alpha$  tends to zero and infinity respectively). The resulting equations which determine the Batchelor limit for the unknown functions are extremely complex for a general helical geometry - they are classified as generalized differential equations for which there is little theory available. The solution of such equations should produce helical vortices of relevance in the study of propeller wakes and swirling flows (such as are produced in the trailing vortex of an airplane), and pipe flows.

Landman has combined his work on laminar-turbulent transition in shear flows with the above work, with the idea that helical waves may be the key to understanding the nonlinear development of viscous flow in a pipe. Circular pipe flow is linearly stable; however it becomes linearly unstable if the pipe is rotated about its axis. He has written a helical Navier-Stokes code, which simulates the time dependent behavior of helical waves in a pipe. When the pipe is rotated, stable steady helical waves which were previously predicted to exist are found to undergo instabilities. Quasi-periodic waves are found, which appear to undergo further bifurcations to irregular (chaotic) time dependence [3]. This behavior seems to arise because of the simultaneous existence of several interacting oscillatory modes corresponding to those predicted by linear stability theory.

Simulations performed when the pipe is stationary have not revealed any finite amplitude helical waves, though these are predicted in a high Reynolds number theory by Smith and Bodonyi [6]. Further simulations will be performed; however it appears that for Reynolds number less than 5000 helical pipe flow is absolutely stable and non-axisymmetric disturbances decay on a fast convective time scale[4].

This evidence refutes any theory of pipe turbulence based on the secondary instability of spiraling waves. Nevertheless it is possible that transition in a pipe is triggered by a small amount of rotation in the inlet flow, implying that the helical waves studied in the rotating pipe, as described above, are relevant in the study of stationary pipe flow.

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# (8) Phase Transitions and Their Stability for Mixed Type Equations (Affouf, Caflisch)

The isothermal fluid equations with viscosity and capillarity terms are

$$v_t - u_x = 0 \quad (1)$$

$$u_t + p(v)_x = \epsilon u_{xx} - \delta v_{xxx}. \quad (2)$$

in which  $v$  is specific volume,  $u$  is velocity,  $x$  is the Lagrangian space variable,  $\epsilon$  is the viscosity coefficient and  $\delta$  is the capillarity coefficient. The capillarity term was first proposed by Korteweg [9] and later analyzed by Felderhof [3] and Bongiorno, Scriven and Davis [2]. This system with a van der Waals pressure law was analyzed by Slemrod [15,16] as a simple model for liquid-gas phase transitions. The van der Waal's pressure term is

$$p(v) = \frac{R\theta}{v-b} - \frac{a}{v^2} \quad (3)$$

for constants  $a, b, R, \theta$ . We use mainly the simplified form  $p(v) = v - v^3$ . For these choices of  $p$ , there is an interval  $[\alpha, \beta]$  of values of the specific volume  $v$  for which  $p'(v) > 0$ . This unstable spinodal region separates the two phases of liquid  $v < \alpha$  and gas  $\beta < v$ . For  $v$  in the interval  $[\alpha, \beta]$  the system (1)-(3) is elliptic in  $x$  and  $t$ , while in the liquid or gas phases it is hyperbolic.

The system (1)-(3) is only a crude model for describing dynamic phase transition, without a solid physical basis. However we believe that the mathematical properties of this system will occur in other more realistic phase transition models [4] or in other conservation laws of mixed type [10,14].

Traveling wave solutions for (1)-(3) representing phase transitions and shock waves were analyzed by Slemrod [15,16] and Hagan and Slemrod [5]. In particular they showed that the capillarity term in (2) is necessary to get a full set of phase transition waves. Moreover, inclusion of capillarity eliminates some phase jumps and thus reduces the multiplicity of solutions for the Riemann problem. These traveling wave solutions were then combined with rarefaction waves by Shearer [11-13] and Hattori [6,7] to produce approximate solutions of the Riemann problem. The rarefaction waves would be exact if  $\epsilon$  and  $\delta$  were zero. By a Riemann problem solution, we mean any combination of traveling waves (shocks or phase transitions) and approximate rarefactions that connect two given states  $(u_l, v_l)$  at  $x = -\infty$  and  $(u_r, v_r)$  at  $x = \infty$ . A surprising result of Shearer [13] is that the Riemann problem solution for given end states  $(u_l, v_l)$  and  $(u_r, v_r)$  is not unique.

In [1] we have performed numerical computations of the initial value problem for (1) and (2), with the pressure law (3), for rarefaction waves and for shock waves and phase transition waves. In particular we numerically showed that all of these waves are stable to sufficiently small initial perturbations, if the end states of the wave are not in the spinodal region. This was surprising because of the non-uniqueness of Riemann problem solutions and because the phase transition waves pass smoothly through the spinodal region. Stability of these phase transitions can be explained by their narrow width. On the other hand, for moderate sized initial perturbations, these waves may be unstable.

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## VII.2 Numerical Methods for Fluid Dynamics

### (9) Convergence of the Point Vortex Method (Goodman, Hou, Lowengrub)

The vortex method for inviscid, incompressible flow is an effective numerical method that introduces no artificial viscosity. Convergence studies of the method required smoothing of the vortices to vortex blobs with large blob size, while point vortices or small blobs were used in practice. Recently Goodman, Hou and Lowengrub [9,10,12] have proved convergence of the point vortex method and the vortex blob method with smoothing of any size. This is an unexpected result that is likely to change our thinking of the way vortex methods work.

The incompressible, inviscid Euler equations in two dimensions in vorticity form are

$$\omega_t + (\mathbf{u} \cdot \nabla) \omega = 0. \quad (1)$$

where  $\mathbf{u}$  is the fluid velocity and  $\omega$  is the vorticity. The vortex method involves the tracking of particle trajectories. We use the flow map, defined by

$$\mathbf{x} : \mathbf{R}^2 \times [0, T] \rightarrow \mathbf{R}^2$$

so that  $\mathbf{x}(\alpha, t)$  is the trajectory of the fluid particle which at time  $t=0$  is at the point  $\alpha$ . The trajectory for fixed  $\alpha$  is computed from the velocity field by

$$\frac{d\mathbf{x}(\alpha, t)}{dt} = \mathbf{u}(\mathbf{x}(\alpha, t), t), \quad (2)$$

$$\mathbf{x}(\alpha, 0) = \alpha. \quad (3)$$

Incompressibility implies that this map is area preserving. The velocity is recovered from the vorticity by the Biot-Savart Law

$$\mathbf{u} = \mathbf{K} * \omega = \int_{\mathbf{R}^2} \mathbf{K}(\mathbf{x} - \mathbf{x}') \omega(\mathbf{x}') d\mathbf{x}' \quad (4)$$

The Biot-Savart Kernel  $\mathbf{K}(\mathbf{x})$  has a singularity of the form  $O(|\mathbf{x}|^{-d+1})$  at the origin, where  $d$  is the number of space dimensions. The vortex method consists in finding approximate particle positions by discretizing (4) and solving the system of ordinary differential equations.

The discretization of (4), however, is tricky due to the singularity of  $K$ . To alleviate this difficulty, Chorin in 1973 introduced the idea of a cut-off kernel,  $K_\delta$ , which is non-singular [7]. Physically, this model, and its subsequent refinements, use blobs of vorticity of a certain shape and with size  $\delta$ , a smoothing parameter, to approximate the vorticity field. This is called the vortex blob method. This method has been shown to be convergent by (e.g. [1,2,3,4,8,11,15]) provided that the blobs are overlapping. In particular, if (4) is discretized by a length scale  $h$  then it is required that  $\delta \gg h$ . Further, the method is higher order accurate depending on the smoothness of the initial vorticity and the type of smoothing done to  $K$ .

In computational practice, one usually takes  $\delta \leq h$  or  $\delta = 0$  [14], and so the previous theory leaves this computational practice in doubt. The point vortex method is exactly the case when  $\delta = 0$ . It corresponds to an approximation of the vorticity field by a collection of point vortices. This method was first proposed by Rosenhead in 1932 to simulate the motion of vortex sheets in two dimensions [16]. The method has the following difficulty, however. Since there is no smoothing, if two point vortices move towards each other, the velocity that each induces on the other becomes unbounded due to the singularity of  $K$ . The convergence of the point vortex method and the vortex blob method for vortex sheets has been demonstrated numerically by Krasny [13] and analytically, for short times, by Caflisch and Lowengrub [5] under the requirement of analyticity. This restriction of analyticity seems to be required because of the ill-posedness of the vortex sheet problem itself. In fact, it was widely thought that the point vortex method would not converge in any finite Sobolev space even for smooth flows. But, recently for smooth flows, Goodman, Hou and Lowengrub have been able to prove the stability and convergence of the point vortex method [9] and the vortex blob method with small blobs ( $\delta \leq h$ ) [10] in two dimensions. In three dimensions, Hou and Lowengrub [12] proved convergence of a point vortex method. These results are valid for arbitrarily long times.

The analysis is based on the framework used in the convergence study for the vortex blob method. The scheme is proved to be stable in  $L^p$  provided that the discretization error is better than first order accurate. Then by a careful analysis which takes into account the oddness of the Biot-Savart kernel  $K$ , the method is shown to be formally second order accurate. Thus, the method is indeed stable and consequently converges for long times. It is important to note that this is only a conditional stability result; losing accuracy may generate instability.

One crucial observation that motivates the stability result is the following: two particles initially separated by a distance  $h$  will remain separated by a distance  $O(h)$  for later times because of the

smoothness of the flow map and the inverse flow map. This suggests that if the point vortex method is better than first order accurate, then any two particle trajectories are separated by a distance  $O(h)$  asymptotically. Therefore, the singular kernel in the point vortex method has a natural cut-off of order  $\delta = O(h)$ . In this sense, the point vortex method is equivalent to the blob method with smoothing size  $\delta = ch$ . This identification allows us to give a particularly simple proof of convergence which relies on the previous theory [12].

The proof of convergence of the vortex method with small blobs essentially follows that of the point vortex method [10]. This is because the vortex method has a natural cutoff  $\delta = O(h)$  which is independent of blobs of size  $\delta \ll h$ . As a consequence, smoothing has little effect if the blob size is small compared to  $h$ . It is also interesting to note that the higher order accuracy of the blob method is lost when the blobs become  $O(h)$  in size or smaller. The method always reduces to second order accuracy in this case.

The convergence proof of the point vortex method in three dimensions also roughly follows the two dimensional proof. There is an additional difficulty in three dimensions, though, which is that the vorticity can be stretched. This problem can be handled using the techniques of Beale and Majda [3,4].

Finally, the theoretical convergence results for two dimensional flows have been confirmed by several numerical experiments [9,10]. The experiments indicate second order accuracy, for a fixed time, as the numerical parameters  $h$  and  $\delta$ , in the case of small blobs, are refined. The numerical experiments were performed on a problem where the exact solution to the Euler equations is known.

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# (10) An Adaptive Vortex Method for Two-dimensional Viscous, Incompressible Flows

(Hou, Papageorgiou, Papanicolaou, Zhu)

J. Zhu [1] introduced in his Ph. D. dissertation an adaptive vortex method as an alternative to the circular vortex, vortex sheet method for simulating numerically two-dimensional, vorticity dominated, incompressible flows at high Reynolds numbers. The basic vortex elements are ellipses with variable orientation and ellipticity. They are convected by the flow, viscosity is simulated by a random walk displacement and new vortex elements are created at the boundaries to simulate vorticity generation. In previous implementations of vortex methods, all vortex elements are circular in the interior and flat (sheets) near the boundaries. By allowing the shape of the vortex elements to change with the flow, we have a more natural representation of the flow field, especially regarding its very different structure near and away from boundaries. T. Hou, G. Papanicolaou and D. Papageorgiou collaborated in this work [2]. T. Hou has done, in addition, a theoretical analysis of a version of the adaptive vortex method [3].

Consider for example a boundary layer flow described by the Prandtl equations. They are derived after stretching the coordinate perpendicular to the boundary and neglecting terms of order  $1/\sqrt{\text{Re}}$  or smaller. Suppose the local boundary is approximated by  $y = 0$  and the vorticity  $\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$  by  $-\frac{\partial u}{\partial y}$ . The vortex method implementation in this approximation is the vortex sheet method, which works well in the regime where the Prandtl equations hold. But the boundary layer is only part of the flow and the rest has to be obtained using the Navier-Stokes equations, which can be done by the circular vortex blob method. This requires matching flows in the two regimes to generate the global solution. Since the nature of a vortex sheet is quite different from that of a vortex blob, the transition in the flow from the boundary layer solution to the solution in the interior is not smooth. With the elliptic vortex element implementation we eliminate this sudden transition by employing a single solution which is good in both regions.

Based on the Prandtl boundary layer approximation, the idea of the elliptic vortex method is this. If we use the same stretching as that in the derivation of the Prandtl equations and keep the term  $\frac{\partial v}{\partial x}$  in the vorticity formulation, the vortex element is going to be an elliptic blob with an order  $\sqrt{\text{Re}}$  ellipticity. This is the fixed elliptic vortex blob method associated with the transformation that is the stretching in the direction perpendicular to the boundary. We extend this idea to general continuously differentiable transformations over the flow region to realize a smooth transition between interior and boundary layer. Given a time-dependent transformation  $\Phi(x, y, t) = (\phi, \psi)$ , the flow region is mapped to a region in a new

coordinate system  $(x^*, y^*)$ . We write the fluid equations in this transformed space and define trajectories of fluid particles, velocity and vorticity. We formulate a vortex method in this transformed system assuming that all the blobs in the new system are circular and that the vorticity is constant inside each blob. We then discretize the kernel which corresponds to the Biot-Savart kernel in the original system. Suppose the blob size is so small that linearization of  $\Phi$  is allowed in the neighborhood of each blob. Then each circular blob in the new system will represent an elliptic blob in the original system. If we have a collection of circular blobs in the transformed space, the transformation  $\Phi$  will furnish us with a collection of elliptic vortex blobs in the original space, each determined by the linear transformation

$$\mathbf{x}^* = \mathbf{x}_i^* + \Phi_{\mathbf{x}}(\mathbf{x}_i, t) \cdot (\mathbf{x} - \mathbf{x}_i)$$

associated with the center of the vortex blob, where  $\mathbf{x}_i$  and  $\mathbf{x}_i^*$  are the positions of  $i$ th blob in original and transformed space respectively, and  $\Phi_{\mathbf{x}}$  is the Jacobian matrix. It remains therefore to solve for a collection of elliptic vortex blobs with constant vorticity inside each blob. We have the analytic solution for the flow induced by each elliptic vortex blob from potential theory and since the equation relating vorticity to velocity is linear, the velocity field associated with the collection of vortex blobs is known everywhere. The updating of vorticity is done in the same way as in the standard method. We move each vortex blob according to the local velocity and a random walk step is added to simulate the diffusion part.

This formulation bridges the gap between the vortex sheet method and the circular vortex blob method in a smooth way, with the full system of Navier-Stokes equations used in all steps. We are required to provide a transformation  $\Phi$  adapted to the flow in some way. Since this transformation is an auxiliary tool designed to insure that the correct equations are solved while implementing the vortex method with as much flexibility as possible, the choice of  $\Phi$  is virtually unrestricted. We have proposed two criteria for  $\Phi$  based on different numerical considerations. The one we use in our calculations is obtained from the real flow by averaging and truncation and satisfies the same boundary condition as the real flow. The other, which we formulated but did not use, determines the transformation by a global minimization of the rate of shear in each time step.

We performed several numerical experiments to test the adaptive vortex method. First we applied it to the flow past a flat plate. Our results compare well with the steady Blasius solution after a sufficiently long time. We also applied this method to the unsteady flow in a channel with a back step. We compared our results both with experimental results and with numerical results using the vortex-blob-sheet method as performed by Sethian and Ghoniem [4]. Good qualitative agreement is obtained in both comparisons.

The adaptive vortex method has several advantages. It does not use the Prandtl equations, so that we can apply it to problems with curved boundaries without having to prescribe a priori length scales associated with the boundary layers, and sharp transitions in the flow are caught in detail. It also avoids using different sets of equations in different regions. We have instead a continuously varying vortex-element representation suitable to specific regions. Since we make no simplification of the Navier-Stokes equations we do not have to worry about applicability of the method to difficult problems where backflow and separation appear. This is evident in the backstep flow problem, since the vortex sheets are not very good at representing the flow in the step corner where the local Reynolds number can be small and the Prandtl approximation therefore inappropriate.

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## (11) A Particle Method for the Numerical Computation of Dilute Fluid-Particle Flows

(Papanicolaou, Zhu)

The continuum modeling of multiphase flows has received a lot of attention by theoreticians, but it is generally agreed that only in very special situations are the continuum equations well-founded and can be used with confidence. One such case is fluid-particle, two-phase flow with the particle phase a dilute or low volume-fraction component. Dilute suspensions of particles in fluids or slightly dusty gases are good examples. A very good survey of theoretical developments in dusty gases is Marble's paper [1].

Papanicolaou and Zhu [2,3] have formulated a continuum model for two-dimensional fluid-particle flow in which the fluid is incompressible and nearly inviscid (high Reynolds number relative to macroscopic length scales) while the particle phase is dilute and the fluid-particle interaction is calculated by the Oseen formula. We assume therefore that the Reynolds number relative to the size of the particles is small. There are no boundaries in the flow region.

To compute such flows numerically we use a vortex-blob method to represent the fluid phase and a particle method to represent the particle phase. The physical particles and the "numerical" particles are very distinct entities, as is always the case with particle methods in numerical computation. Modeling of the interaction between the two phases at the numerical level is the main difficulty that must be faced in the problem. We have done extensive numerical experiments to test the performance of various interaction models. We have found that the use of Voronoi diagrams both in the fluid and particle phase is very effective. The necessary interpolations are done by an algorithm [4] that is suitably local. Extensive numerical testing has shown that our vortex-particle method produces good results for flows in the regime described above.

Previous computations of fluid-particle flows at high Reynolds numbers using the vortex method [5,6] treated the particle phase as tracer particles advected by the fluid but not affecting it by their motion.

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## (12) Lagrangian Finite Element Method (Chacon, Hou)

Tomas Chacon and Thomas Hou [1] have proposed a new Lagrangian finite element method to approximate the 2-D incompressible Euler equations. In this method, they transport the initial triangulation by streamlines of the flow and approximate the vorticity by piecewise linear finite elements. As a consequence the vorticity is exactly computed on the mesh points of the triangulation. Their method is conceptually similar to the vortex method, except that they do not need to introduce smoothing for the singular kernel. This is possible because the product of the Biot-Savart kernel and a polynomial can be integrated analytically over any triangle. This enables them to obtain a stable and consistent discretization for the singular velocity integral without smoothing. The kind of stability result they obtain here is a conditional one in that the method is stable if the consistency error is better than first order accurate. In order to ensure stability, they need to use linear or higher order finite element approximation for the vorticity field and a better than first order time integration formula. Then the Lagrangian triangulation will not become degenerate provided  $h \leq h_0(T)$  is small. Consequently the method is stable and convergent. Numerical experiments have shown that the method converges with second order accuracy for long times.

The analytical techniques used here are of independent interest, since they do not need to differentiate the singular kernel in proving the stability as is the case of the vortex method. Only the smoothness of the solution and the  $L^1_{loc}$  property of the kernel are used in the stability argument. This enables them to prove convergence of the method for a more singular kernel than that of the 2-D Euler equations [3]. Another application of this method is in calculating the motion of vortex patches. The fact that the kernel is integrated exactly over each triangle makes the finite element method very natural for approximating vortex patches. Numerical computations have shown that the method provides accurate approximations for two interacting elliptical patches up to the time when the patches are nearly singular. A convergence analysis for vortex patch problems is in progress [2].

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### VII.3 Random Media and Composite Materials

#### (13) Direct and Inverse Problems for Pulse Reflection of Waves from Random Media

(Asch, Kohler, Papanicolaou, Postel, Sheng, White)

Consider a pulsed acoustic, electromagnetic or elastic wave impinging on a material that is inhomogeneous, the earth's crust for example. The reflected signal is measured at the surface and it looks very noisy, like a seismogram. What information about the medium can we extract from this signal? In reflection seismology this is a classic question that has received a great deal of attention over the last forty years. It has also received a lot of attention as a difficult problem in mathematics: the inverse scattering problem. George Papanicolaou in collaboration with R. Burridge (now at Schlumberger-Doll Research), P. Sheng and B. White (at Exxon Research and Engineering Co.) and more recently W. Kohler (Virginia Polytechnic Inst.), M. Postel (Courant Institute) and M. Asch (a graduate student at the Courant Institute) began a study of this question as a problem in stochastic processes. We model the reflecting material as a random medium, characterize all possible reflected signals as a particular class of nonstationary stochastic processes and then pose the recovery of material properties as a statistical estimation problem for this class of processes. We test the theory by generating numerically reflected signals and then using our estimation algorithm on them. References [1-10] contain our work. Reference [11] is a recent textbook on seismology where a variety of inverse problems are posed and references [12-13] are the papers that gave us the idea to start this research. They contain numerical simulations for pulse reflection from random media but they do not address the inversion questions.

How can we develop a theory for reflected signals that is realistic but simple enough to be useful? First we must choose carefully a model for the reflecting medium and then analyze its scattering properties with methods that are well suited to the model. We have focused attention on layered acoustic half spaces so far. Locally layered elastic half spaces can also be studied by our methods and they are rather realistic models for the earth. The next issue is modeling of the inhomogeneities, which means that we must decide what form the density and bulk modulus of the material are to have. We have already assumed that they are functions of one variable only: a layered medium. Motivated by experience with other problems in materials science, we decided to model the medium properties by random functions that vary on two length scales. Their mean values vary on a macroscopic scale, for example kilometers in the earth model. Their fluctuations are rapidly varying but not necessarily small. In the earth model the

speed of sound varies on the scale of two or three meters by as much as thirty percent, as we know from well log data. Of course there are variations on many other scales in the earth but as long as there are two well separated scales that predominate, our methods of analysis apply. The final step in the modeling is the choice of the probing pulse that will generate the reflected signal. We choose the pulse width to be on an intermediate scale between the macro and microscales. In the earth model the pulse width may be 150 meters (a 20-40 Hz pulse) which is narrow compared to the 10 kilometer depth we want to probe but broad compared to the 3 meter width of a typical layer.

Now the problem is formulated as one in the asymptotic analysis of stochastic equations on which we have worked extensively in the past and have developed mathematical techniques. The main outcome of the asymptotic theory is a precise characterization of the reflected signals as a relatively simple class of nonstationary Gaussian processes that depend primarily on the macroscopic properties of the medium and are practically independent of the small scale variations.

Once the scope of the theory is fully understood we can go to the inverse question which is an unusual estimation problem for a Gaussian process. The main advantage of our theory is that we recover directly the large scale properties of the medium and never get tangled with reconstructing all its details. This is the main drawback of other methods, overcome mainly by ad hoc "processing" of the reflected signals.

To implement the theory we must first have an efficient way to generate numerically reflected signals for our model. This has been done for the earth model with the parameters given above when a plane wave pulse is incident. A finite difference method is used. To resolve the small scales (3 meters) we use a spatial mesh that is about 30 cm. Then we solve for long enough times that a signal can go down 8-10 kilometers and return. This takes about 150,000 time iterations when the speed is 3 km/sec on average. A well-vectorized code on the cyber 205 produces a reflected signal with these parameters in about 25 cpu minutes.

Once we have good reflected signals (synthetic seismograms) we can try our inversion algorithm. This was done in detail in [4] and the results are quite good if enough redundancy is used. That is, if we use enough independent samples of reflected signals in our estimation algorithm. The quantity estimated is the smooth (averaged) part of the local sound speed. For a good estimation we need about 50 samples. But in reality we have only one reflected signal. How do we generate other realizations? In reflection seismology the exciting pulse is not a plane wave; it is also localized in space because it comes from a

pulsed point source. We then measure reflected signals at different points on the surface and treat them as separate realizations. This must of course be done correctly and the theory to do it is in [5]. The main point of our computational research in the immediate future is the numerical simulation of such reflected signals.

Generating reflected signals from a pulsed point source over a randomly layered medium is a very big computational task. For a plane wave we used a spatial resolution of 30 cm in a finite difference scheme. In a two or three dimensional setting such resolution is out of the question when we want propagation over kilometers. In dimensions higher than one we may use a layer-specific method for computing the wave without having to resolve it inside the layers. This requires taking Fourier transforms in space horizontally and in time with enough resolution to capture the space-time fluctuations in the signal. If we use 2000 layers (6 km maximum depth) it becomes clear that this is again a very big problem, even for generating one realization. We have at present a well-vectorized code for this problem (layer algorithm plus Fourier and Fourier-Hankel transforms) running quite efficiently on an ETA 10. It is producing very accurate results but it still takes about 30 cpu minutes on an ETA 10 to generate one-second signals at about 20 surface locations. We are also developing on an IRIS 4D/50 GT the graphics necessary to visualize the results. We would like to be able to generate 3-4 second reflected signals at 50 to 100 points on the surface in different spatial configurations so that we can test our theory and, more important, test various inversion schemes.

W. Kohler, G. Papanicolaou and B. White are completing a monograph on the theory that has been developed, which will be published by North Holland in 1989.

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**(14) Reflection and Transmission of Gravity Waves in a Channel with Irregular Bottom Topography (Nachbin, Papanicolaou)**

The propagation of long gravity waves in a channel can be analyzed using the shallow water equations. For weakly nonlinear waves traveling in one direction this is the Korteweg-deVries equation. For fully nonlinear waves traveling in both directions one can use conservation laws analogous to those in gas dynamics, having a barotropic equation of state with  $\gamma=2$ . For small amplitude, linear waves, the shallow water equations reduce to the acoustic equations [1].

The shallow water equations are only valid, however, when the variations in the bottom topography are slow compared to the average depth. In [2] we showed that if the bottom topography varies periodically with the period comparable to the depth and if weakly nonlinear, long waves propagate in one direction, then again the Korteweg-deVries equation is a valid approximation. The coefficients in it have to be determined by solving cell problems as is usual in homogenization [3]. The analysis of [2] does not generalize to bottom topographies that vary randomly. We expect substantial reflection in the random case, which means that the assumption of waves moving in one direction, underlying the Korteweg-deVries approximation, is no longer valid. Furthermore, even in the linear case there is no clear understanding of how to treat reflection and transmission in the random case [4-7].

A. Nachbin and G. Papanicolaou [8,9] have begun a systematic analysis of the reflection-transmission problem with random bottom topography. The analysis is in two parts and is restricted to linear problems at present. The first part is the development of an efficient code for calculating numerically the reflection and transmission of wave pulses by random bottom inhomogeneities. We use a boundary element method as has been done previously by several investigators [10-16]. In previous implementations the boundary element method has rendered dispersive effects better than finite difference methods of comparable complexity. There is a simple theoretical reason for this that we have clarified. Apart from that our implementation of the boundary integral method differs from previous ones only in that our code is fully vectorized and runs at the equivalent of about 400 megaflops on the ETA 10 at the John von Neumann Center at Princeton. This allows us to study propagation of pulses for a long enough time that multiple scattering from the bottom irregularities is important. It is also very important for the next step in our study, the study of nonlinear effects.

The second part of our study is theoretical. We use the asymptotic theory of stochastic equations to analyze reflection and transmission. We have shown that although the acoustic approximation is not valid

for long waves over an irregular bottom, the statistical description of the reflection-transmission process has an equivalent acoustic interpretation, with the dispersive effects accounted correctly. An important feature of our analysis, which parallels that of [17], is that the bottom inhomogeneities need not be small. It also allows us to address inverse problems: can the reflected surface waves tell us what the large-scale features of the bottom topography are? This question has a rather complete answer. Naturally the additional length scale present in this problem, the average depth, not present in the acoustic case, complicated the implementation of the theory considerably.

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(15) Bounds on the Effective Properties of Composite Materials (Avellaneda, Kohn, Milton)

Despite a surge of interest in composite materials it is typically not known how to calculate the range of possible effective conductivity tensors as the microstructure is varied over all configurations. This question is relevant to optimization. For isotropic poly-crystalline materials constructed from a given single uniaxial crystal, Schulgasser [1] made a conjecture in 1982 that the most resistive configuration would be a sphere assemblage in which the crystal axis is directed radially outwards in each sphere. Avellaneda, Cherkaev, Lurie and Milton [2] have now proved this conjecture, using a technique which involves null-Lagrangians. The technique is evidently powerful since it also enabled us to prove a sharp phase interchange inequality which had been conjectured in 1982 [3]. This inequality

$$\sigma^* \tilde{\sigma}^* / \sigma_1 \sigma_2 + (\sigma^* + \tilde{\sigma}^*) / (\sigma_1 + \sigma_2) \geq 2$$

correlates the effective conductivity  $\sigma^*$  of an isotropic composite, made from two isotropic phases of conductivities  $\sigma_1$  and  $\sigma_2$  with the effective conductivity  $\tilde{\sigma}^*$  having the same microstructure but with the phases interchanged. This inequality has several important practical implications [4].

Another technique to bound the range of possible effective tensors is to use Hashin-Shtrikman variational principles. Milton and Kohn [5] have streamlined this method and used it to obtain bounds on the elasticity tensor of two-component materials. The resulting bounds are natural generalizations to anisotropic composites, of the widely known and practically important Hashin-Shtrikman bounds. The bounds are optimal and are attained with the class of sequentially layered laminate materials. Such materials are constructed by layering together the components on widely separated length scales.

Milton has also investigated [6] the problem of conduction in two-dimensional, two component materials in which the conductivity tensor is asymmetric, as it will be in the presence of a magnetic field. By the use of a suitable transformation he finds this problem is isomorphic to one in which both components have symmetric conductivity tensors, i.e. in which the magnetic field is absent. This surprising result enables him to completely characterize the set of all possible asymmetric effective tensors of the composite as the microstructure is varied.

Milton plans to continue characterizing the set of possible effective conductivity tensors associated with composites of various given materials. The problem bears some similarity with the study of phase transitions in Statistical Physics. We need to identify microstructures, i.e. phases, that minimize or maximize some linear combination of energies. These energies may for example represent the electrical energy stored in the composite when the field is applied in several different test directions. In a typical

problem "microstructure" transitions need to be identified, and some physical explanation for the transition needs to be given. This study is still in its infancy. No one yet knows what are the possible types of transitions that can occur.

The available methods for bounding effective constants, all seem to have their limitations. In particular the methods fail to give a complete characterization of the set of possible effective tensors when an isotropic material is mixed with an anisotropic material in prescribed proportions even in two-dimensions. The problem is generic and stems from a lack of knowledge of the extent to which electric fields can be concentrated in small volumes in the composite. A related question is to investigate the probability distribution of field intensities in a composite. While this is an intractable problem for most materials, one can hope to obtain results for a special class of hierarchical models. These models are interesting because they exactly realize [7,8] the well-known effective medium approximation for the conductivity.

It may also be possible to further explore the connection between various analytic representations for the effective conductivity as a function of the component conductivities in a multi-component composite. From the analytic properties of this function Golden and Papanicolaou [9] have deduced an integral representation for the effective conductivity function. It now seems their representation can be modified to incorporate the natural symmetries amongst the roles of the components. This should lead to a very elegant representation formula. On the other hand there is a continued fraction representation for the effective conductivity [10] that was obtained by recursion methods in the Hilbert space in which the conductivity problem is set. It remains to connect these representations and to find the analytical significance of the parameters that enter the continued fraction expansion. This problem, if solved, would likely have a profound influence in the study of functions of several complex variables.

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# (16) Optimality of Bounds for the Polycrystalline Problem (Milton, Nesi)

Composite materials have received a lot of mathematical attention in the last few years. This is in part due to their relation to the calculus of variations, optimal control theory and homogenization of families of elliptic p.d.e.'s with rapidly oscillating coefficients.

A particular goal of much recent work is the so called "G-closure problem", which seeks to determine the set of all possible composites attainable from a given class of component materials. The thesis of Nesi [12] is concerned with this problem in the context of polycrystalline composites, for electric conductivity in three space dimensions.

Polycrystals are composite materials consisting of individual grains (or crystallites) having the same basic conductivity tensor but varying in orientation from one to the next. The effective conductivity of such a material has been investigated by many authors including [1-6].

From the mathematical viewpoint, one can write the field equations for the electric field  $\vec{e}$  and the current field  $\vec{j}$  in a polycrystalline composite in the form

$$\vec{j}(\vec{x}) = \hat{\sigma}(\vec{x}) \vec{e}(\vec{x}), \quad \vec{e}(\vec{x}) = -\text{grad } u(\vec{x}), \quad \text{div } \vec{j}(\vec{x}) = 0$$

where  $\hat{\sigma}(\vec{x})$  is the local conductivity tensor and  $u(\vec{x})$  is the electrostatic potential. By the definition of a polycrystalline composite,  $\hat{\sigma}(\vec{x})$  has the form

$$\hat{\sigma}(\vec{x}) = R^T(\vec{x}) \sigma R(\vec{x}),$$

where  $\sigma = \text{diag}\{\sigma_1, \sigma_2, \sigma_3\}$  is nonnegative definite and represents the conductivity tensor of the basic crystallite and  $R(\vec{x})$  is a field of orthogonal matrices in  $R^3$  reflecting the variation of crystal orientation in the phase geometry. One way to define the effective conductivity tensor  $\sigma^*$  is

$$\vec{v}^T \sigma^* \vec{v} = \lim_{\varepsilon \rightarrow 0} \inf_{u(\vec{x})} \frac{1}{|D|} \int_D (\nabla u(\vec{x}))^T \hat{\sigma}\left(\frac{\vec{x}}{\varepsilon}\right) \nabla u(\vec{x}) d\vec{x}$$

$$u(\vec{x}) = \vec{v} \cdot \vec{x} \quad \text{on } \partial D.$$

Significant progress in understanding the effective moduli of polycrystalline composites has recently been obtained in [7]. That work derives improved bounds on the effective conductivity using a method based on null Lagrangians, c.f. [8].

The goal of Nesi's work is to explore the attainability of a certain "lower bound" proved in [7] for the effective conductivity of a polycrystalline composite. His main result is that this lower bound is attained by a large class of materials. When specialized to isotropic effective conductivities, his result is

as follows: given a basic crystal with conductivity tensor  $\sigma = \text{diag}\{\sigma_1, \sigma_2, \sigma_3\}$ , the lowest possible isotropic effective conductivity of a polycrystalline composite made from this material is

$$\sigma^* = \sigma_s \quad (1)$$

where  $\sigma_s$  is the unique positive root of the cubic equation

$$\sigma_s^3 + \sigma_s^2(\sigma_1 + \sigma_2 + \sigma_3) - 4\sigma_1\sigma_2\sigma_3 = 0. \quad (2)$$

His other results are, in a sense, extensions of (2) to the case of anisotropic effective tensors. He proves that the bound

$$\sigma_s^3 + \sigma_s^2(\sigma_1^* + \sigma_2^* + \sigma_3^*) - 4\sigma_1^*\sigma_2^*\sigma_3^* \leq 0. \quad (3)$$

is saturated (i.e. equality holds) for a large class of "sequentially laminated" materials, corresponding to a variety of different anisotropic effective conductivity tensors. Before his work the attainability of (1), was known only in the very special case when the basic crystal has uniaxial symmetry (i.e.  $\sigma_1 = \sigma_2$  or  $\sigma_2 = \sigma_3$ ). Nothing was known in the general case  $\sigma_1 < \sigma_2 < \sigma_3$ .

To obtain such a result, one must find a microgeometry for which the associated effective conductivity achieves equality in the bound (3). He has found several different constructions which do this. All are based on sequential lamination. This is a very flexible method for constructing microgeometries with explicitly computable effective moduli; it has been successfully applied to many G-closure problems, see e.g.[11]. The problem with the method of sequential lamination is that there are too many parameters to be chosen. In a certain sense, Nesi's contribution is to show what are the extremal ways of selecting these parameters in this problem.

The first step, initiated by G.W. Milton, was a set of numerical experiments: they looked for an isotropic material with the lowest possible conductivity by laminating the basic crystal with some rotation of itself. By examining the extremal microgeometries generated numerically, they deduced which strategy of lamination seemed to be the most efficient. Then Nesi proved analytically the attainability of the lower bound (3) by introducing an increasing number of length scales (in the context of reiterated homogenization, see [9] and [10]) and making use of a classical fixed point theorem. The constructions Nesi introduced give rise to microgeometries which were never considered before in the literature. This type of microgeometry is called an infinite rank composite material, since the homogenization has to be reiterated infinitely many times to get the extremal composite which attains the bound.

Nesi has also studied schemes which correspond to finite rank lamination. Such schemes depend on a smaller number of parameters which, as they vary, lead to trajectories in the space of effective tensors, lying on the boundary of the G-closure. He has obtained a complete characterization of those laminations which are, in some sense, extremal. Nesi also compares the finite and infinite rank schemes and shows, in particular, that the latter is more powerful.

These results on attainability strongly suggest that some bounds on the effective moduli of polycrystalline composites remain to be discovered. Such new bounds are now being sought, by trying to make use of additional null-Lagrangians besides those used in [7].

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### (17) Iterated Homogenization and Differential Effective Medium Theory

(Avellaneda, Milton, Papanicolaou)

In 1935 Bruggeman [1] introduced a class of dielectrics for which the effective or macroscopic properties could be computed exactly from a relatively simple formula. His theory came to be known as the "Effective Medium Theory" and is used extensively in a variety of applications. Bruggeman described the class of materials for which his theory was valid in a qualitative way: they have to contain inhomogeneities with arbitrarily small scales, and the addition of inhomogeneities of one scale can be dealt with incrementally, as if all inhomogeneities at coarser scales were homogenized or had only macroscopic effects. This almost axiomatic description of the materials, followed by the formula for their effective dielectric constant with virtually no other computations, was a veritable tour de force by a brilliant physicist. But because the formula that he postulated is a simple one and because it can also be derived (as was done in the 1960's) by a self-consistent scheme, confusion arose as to precisely what class of materials has this effective behavior. More mathematically inclined researchers can go a step further and question the existence of real materials with Bruggeman's effective behavior altogether.

Milton [2] first showed that Bruggeman's theory (called the coherent potential approximation in the meantime) is realizable. His proof is complicated and leaves the reader with the general impression that the class of materials for which the theory works is small. Norris [3] gives a contemporary account of effective medium theory, including the differential effective medium theory in which the volume occupied by the inhomogeneities of the various scales need not be all the material volume. The conceptual framework is a more qualitative version of Bruggeman's rules or "axioms" but the issue of what is exactly the underlying material is not addressed.

Avellaneda [4] gives a very simple and rather complete mathematical description of effective medium theory starting with real materials and using the theory of iterated homogenization [5]. Bruggeman's "axioms" are realized nicely, as they were in Milton's work [2], and it is now clear that a very broad class of materials, called hierarchical materials, can be described by effective medium theory and its generalizations. Iterated homogenization exploits very efficiently the separation of scales present in the hierarchical materials. The ideas here are closely connected with the renormalization group formalism where the separation of scales is done in a heuristic way. In fact the renormalization group formalism applied to composite dielectrics reduces to iterated homogenization if separation of scales is imposed. The difficult, deep and open question here is why the renormalization group works even when there is no clear separation of scales.

In [6] Avellaneda and Papanicolaou analyze a class of nonlinear composites with hierarchical structure and generalize the effective medium theory to fully nonlinear materials. Nonlinear iterated homogenization is used and both the symmetric (Bruggeman) version as well as the differential effective medium theory are studied.

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## (18) Numerical Approximations of Oscillatory Solutions to Partial Differential Equations

(Avellaneda, Hou, Papanicolaou)

There are many computational problems with highly oscillatory solutions, e.g. computation of high frequency acoustic and electro-magnetic fields, properties of composite materials and turbulent flow. Sometimes the original equations describing this problems can be replaced by effective equations modeling some average quantity without the oscillations. Geometrical optics, homogenization of composite materials and turbulence models are such examples [3].

Whenever effective equations are applicable, they are also very useful for computational purposes. However, there are many situations for which we do not have well-posed effective equations or for which the solution contains different frequencies such that effective equations are not practical. In these cases, the question naturally arises whether the original formulation can be used to obtain useful approximations without accurately resolving small scale structures on the computational grid.

In this project, we investigate the possibility of approximating original equations directly without resolving all scales in the solutions. The classes of equations we consider include semilinear hyperbolic systems, elliptic problems and incompressible Euler equations with oscillatory solutions. In particular, we study convergence properties of several numerical methods, i.e. finite difference methods, particle methods and random choice methods, when applied to these problems with oscillatory solutions.

There are three main sources of difficulties for discrete approximations of highly oscillatory solutions.

(i) The first is the sampling of the computational mesh points ( $x_j = j \Delta x$ ,  $j=0,1,\dots$ ). There is the risk of resonance between the mesh points and the oscillations. If, e.g.  $\Delta x$  equals the wave length of the oscillation, the discrete initial data may only get values from the peaks of the oscillatory solution. We can never expect convergence in that case. Thus  $\Delta x$  cannot be completely independent of the wave length [8-10].

(ii) Another difficulty comes from the approximations of advection. The group velocity for the differential equation and the corresponding discretization are often very different. This means that an oscillatory pulse which is not well resolved, is not transported correctly by the approximation. Furthermore, dissipative schemes rarely advect oscillations correctly. The oscillations are damped out very fast [8,11].

(iii) Finally, the nonlinear interaction of different high frequency components in a solution must be modeled correctly. High frequency interaction may produce lower frequencies that influence the

averaged solution [8,13,16].

Bjorn Engquist was the first to study this problem. He showed that for the semilinear Carleman equations [5] with oscillatory initial data, a particle method converges essentially independent of the oscillatory wave length (say  $\epsilon$ ) [8]. Convergence essentially independent of  $\epsilon$  means that the error is small for all samplings of the grid outside a set of arbitrarily small measure [8].

Tom Hou continued the study of this problem in his Ph.D. thesis under the supervision of Engquist [14]. They chose the semilinear discrete Boltzmann equations in kinetic theory as a model problem [15]. In particular, they consider the Broadwell model whose properties are very different from those of the Carleman model [4]. In such equations, the high frequency components can also be transformed to the lower frequencies through the nonlinear interaction, which depend sensitively on the velocity components [13,16]. Their study shows that particle methods can provide a good approximation to highly oscillatory solutions of the discrete Boltzmann equations. The crucial step in their particle method is the updating of the numerical solutions along their characteristics without using local interpolation. This guarantees that the advection step is correctly approximated without introducing numerical dispersion or dissipation. Moreover, since the oscillations in the solutions are propagated along characteristics, the particle method approximation captures the nonlinear interactions properly. Thus if enough ergodic mixing is provided in the samplings of the grid, classical ergodic theory implies that the numerical solution will converge to the continuous averaged solution with probability one [1,8,9,10]. That is, the set containing bad samplings has arbitrarily small measure.

Another application is the vortex method [6] for 2-D incompressible Euler equations with highly oscillatory vorticity. Although many numerical experiments seem to suggest that vortex methods could produce useful results for high Reynolds number flow without resolving the small scales in the solution (see e.g. [17]), there is still lack of theoretical justification for these observations. For our model problem with oscillatory but bounded vorticity field, we can show that the vortex method converges to the homogenized solutions of the Euler equations essentially independent of the oscillation [7,14]. The analysis here follows from similar observations for the discrete Boltzmann equations.

There are problems with oscillatory solutions for which sampling of the grid alone is not enough to guarantee convergence of the numerical method essentially independent of the oscillation. For second order elliptic equations with oscillatory coefficients, Tom Hou in collaboration with Marco Avellaneda and George Papanicolaou shows that except in one dimensional cases, finite difference methods in general will not converge to the correct homogenized solutions with probability one independent of

oscillations [2]. Certain small scale features in the solution have to be resolved well by the computational grid. On the other hand, it is still possible to obtain useful approximations without resolving all the small scale features in the solutions. In the case of periodic coefficients, we show that it is enough to require  $(h/\epsilon - [h/\epsilon])$  to be small, where  $h$  is the grid size,  $\epsilon$  is the oscillatory wavelength and  $[h/\epsilon]$  is the integer part of  $h/\epsilon$  [2].

In the near future we would like to extend the sampling idea to problems in fluid dynamics where the oscillations are produced by the equations themselves. This leads to the well-known subgrid modeling problem for which homogenization ideas are only marginally useful [18] at present.

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### **(19) Recent Progress in Traveltime Tomography (Berryman)**

One recent innovation in traveltime tomography is the introduction of variational methods to aid in the construction of the initially "close" models of structure required for the success of diffraction tomography in all but the simplest situations. The essential idea is to construct a variational functional based on the physics of wave propagation through elastic materials, i.e., the minimum of the variational functional occurs when the waves satisfy the correct partial differential equations [1,2]. The tomographic data obtained from all source and receiver pairs then form a complicated set of constraints on the variation which are easily treated on the same footing as the wave equations. The advantage of this approach is that, instead of solving the partial differential equations for the current (approximate) model exactly everywhere at each step of an iteration process, variational methods can be used to obtain reasonable approximate solutions quickly. For example, there is clearly not much point in satisfying Snell's law exactly at every interface for a model structure whose chosen cell pattern is arbitrary and whose wave speeds are initially expected to be a poor approximation to the true values; in particular, since the locations of interfaces between cells are generally chosen in an arbitrary manner, the choices made should not have a large effect on the final result, but with common methods often do. The new approach that has been developed by Berryman [3-5] constructs rays that satisfy Fermat's principle globally and approximately satisfy Snell's law locally at each stage of the calculation and with a level of accuracy consistent with that of the model. The method has been implemented in two-dimensions by Berryman and Yorkey [6] using a new, simple, and fast simplex algorithm [7] for finding optimum coefficients in the Fourier sine series expansion of the graph of the ray. This ray tracing method is currently in use in our traveltime reconstructions, and provides an efficient means of generating the "close" model of structure required as input to the sophisticated methods of Devaney, which are expected to converge more rapidly once they can be applied. In some applications, the model obtained from traveltime tomography may provide a satisfactory estimate of the wave speed structure by itself, without needing the additional data and effort required for diffraction tomography.

New algorithms have recently been developed for inverting traveltime data in media with high wave speed contrasts (and therefore with strong ray bending) [3,4]. One of the most significant new contributions of this work has been the realization that Fermat's principle (i.e., the true ray path for the first arrival is the one of least traveltime) makes a very powerful statement about the feasibility or lack of feasibility of trial rays through a given wave speed model for the medium of interest [4-6]. Nonfeasible rays are trial rays with traveltimes shorter than that measured -- such rays are physically impossible.

Feasible rays are all other rays with traveltimes as long as or longer than that measured. Fermat's principle may therefore be used to "convexify" the fully nonlinear inversion problem. Since there exists a definite convex region of the slowness model space -- depending only on the measured traveltimes -- that contains all feasible models, the "solution" (if one exists) to the traveltime inversion problem lies on the boundary of this convex set. Thus, traveltime tomography can be reduced to a problem in nonlinear constrained optimization. Obtaining an exact solution of this optimization problem would be very hard indeed because the pertinent vector space is multidimensional and, although every point on the feasibility surface can be found in principle, the computations involved in mapping such a multidimensional surface are prohibitive. However, our knowledge of the existence of this feasibility surface is sufficient to provide a simple, approximate figure of merit that allows us to stabilize the reconstruction algorithm with a negligible increase in the computation time.

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## VII.4 Nonlinear Optics

### (20) The Focusing Singularity of the Nonlinear Schroedinger Equation

(Landman, LeMesurier, Papanicolaou, Sulem, Sulem)

The nonlinear Schroedinger equation (NLS)

$$i \frac{\partial \psi}{\partial t} + \Delta \psi + |\psi|^2 \psi = 0$$

$$\psi(0, \mathbf{x}) = \psi_0(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^N$$

arises frequently in the description of wave phenomena as envelope equation for field amplitudes. In nonlinear optics it describes approximately the propagation of narrow beams in a material whose index of refraction increases or decreases with field intensity leading to focusing or defocusing of the beam. Here the variable  $t$  represents distance along the direction of the beam axis and the spatial variables  $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$  are coordinates in the cross sectional plane of the beam. In plasma physics it arises with  $N = 3$  and it describes the evolution of Langmuir waves in plasmas, in the subsonic limit. Recent surveys of applications and basic properties of the nonlinear Schroedinger equation are [1,2].

The main mathematical problem associated with NLS is the existence of singular solutions in two or more dimensions ( $N \geq 2$ ), their local form and the nature of the solutions (if they exist) past the singularity. The singularity occurs when the nonlinear focusing overcomes wave dispersion, the beam amplitude becomes infinite on its axis and its cross section shrinks to zero. These basic facts have been known for over twenty years [1,2] but the local form of the singularity is difficult to determine, analytically or numerically, and several attempts in the literature proved later to be contradictory. Understanding of the local form of the singular solutions is important for many reasons. Mathematically it is a problem comparable to understanding the structure of discontinuous solutions of model conservation laws such as Burger's equation or the Korteweg-De Vries equation in the zero dispersion limit: it is a canonical situation in which competing effects (focusing nonlinearity and wave dispersion) produce phenomena that appear quite generally in much more complex systems. Physically and numerically it is an equation on which, because of its relative simplicity, one can develop and study expansions, computational schemes, etc. that can be used in more elaborate settings: saturated nonlinearities (where  $|\psi|^2$  is replaced by  $|\psi|^2(1 + |\psi|^2)^{-1}$ ) and effects of backscattering in beams, the Zakharov equations in plasmas and so on.

George Papanicolaou in collaboration with D. McLaughlin (University of Arizona) begun a study of the local form of the focusing singularity in 1980. Little progress was made until 1984 when in collaboration with B. LeMesurier (then a graduate student, now at the University of Arizona), C. Sulem and P.L. Sulem (from France and Israel) extensive numerical computations were conducted. Many numerical computations of singular solutions of NLS had of course been done by other investigators previously, but they did not have the resolution near the singularity to distinguish between competing theoretical constructions based on asymptotic analysis. The numerical computation of singular solutions of an equation in which there are no dissipative effects whatsoever, such as NLS or a wave equation, is exceedingly difficult. In [3] we introduced a method, we call it **dynamic rescaling**, which allowed us to compute numerically singular solutions up to amplification of order  $10^{12}$ . The idea is to let the size of the numerical mesh depend on time and to derive an equation for it that has to be solved along with NLS (suitably rescaled). An equation for the mesh-scale factor is obtained by demanding that a suitable norm of the wave function stays bounded. The new coupled system (NLS plus the mesh-scale factor equation) has no singularities and therefore can be solved by standard numerical methods. We used two different numerical methods to check our results. The nature of the focusing singularity can be read off from the time evolution of the mesh-scale factor which is computed very accurately.

In [4,5] we presented more extensive calculations. We concluded that in dimensions  $N > 2$ , the supercritical case, solutions blow up like  $(t^* - t)^{-1/2}$ , where  $t^*$  is the time where the solution becomes singular, and we characterized the spatial profile of singular solutions through a nonlinear eigenvalue problem. We have not succeeded up to now in providing a full theory for this difficult mathematical problem, but we have obtained a lot of information via formal asymptotics and numerical computations. In two dimensions  $N = 2$ , the physically interesting case in nonlinear optics, our computations showed that the blow-up rate  $(t^* - t)^{-2/3}$ , predicted by Zaharov and Synach [8], or  $[(t^* - t)/\log(t^* - t)]^{-1/2}$ , predicted by Talanov [9] and Wood [10], are not correct. We were not able however, even with our computations, to guess the true form of the blow-up rate. In [4,5] we included many rather extensive calculations of solutions of NLS with saturated nonlinearities. They have many intriguing features, such as dispersive damping after repeated focusings, that we do not understand analytically. These calculations were done on a Cray XM-P. The calculation of the singular solutions without saturation was done on our own mini computers (mostly the Celerity 1200).

By the summer of 1987 we were convinced that no amount of numerical computation, no matter how elaborate or accurate, could help us determine the form of the focusing singularity in two dimensions

$N = 2$ , the critical case. Using a matched asymptotic expansion we derived in [5] a solution that was singular in the same way Zaharov and Synakh [8] predicted, by very different methods. In reworking our analysis we realized that our asymptotic solutions (and those of [8]) had a very different profile than the one that the numerical computations produced. There was no way to fix this except by perturbing the dimension to  $N > 2$  and then using the profiles from the nonlinear eigenvalue problem of the supercritical case. Those profiles were much closer to the numerically observed profiles when  $N > 2$  was near 2. With this insight, and in collaboration with M. Landman, we were able to determine completely [6,7] the singular blow-up in the critical case which is  $[(t^* - t)/\log \log(t^* - t)^{-1}]^{-1/2}$ . This also explains why it is difficult to get such information from numerical calculations.

We are currently working on an extension of the dynamic rescaling ideas to more general situations where there are multiple peaks in the solution, the peaks move and they need not have (locally) isotropic profiles. Our aim is to write a general code in three space dimensions in which we implement these ideas. We are also studying intensively, with X. Wang (a graduate student), the mathematical questions that arise in connection with the nonlinear eigenvalue problem for the singular profiles.

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## (21) Nonlinearity and Localization in One Dimensional Random Media

(Knapp, Papanicolaou, White)

R. Knapp, G. Papanicolaou and B. White [1] have analyzed the combined effects of nonlinearity and randomness in the propagation of time harmonic waves in one dimensional media. In nonlinear optical media the index of refraction depends on the intensity of light, giving rise to many interesting effects, including optical bistability. When an optical device can have two different output states for a given input intensity (depending on hysteresis) it is said to be bistable. On the other hand in linear media with random inhomogeneities, different phenomena arise, such as localization. When localization occurs the transmitted intensity decays exponentially as a function of the length of material the light passes through.

Optical bistability was first observed experimentally in a Fabry-Perot etalon. The etalon is constructed from a slab of nonlinear material of thickness  $L$  sandwiched between two partially reflecting dielectric mirrors. Randomness can be introduced either from impurities or by constructing the medium with alternating films of nonlinear and linear material of random thickness. We consider a time harmonic model for the etalon with a wave incident from the right

$$u_{xx} + k^2(1 + \epsilon\mu(x) + w|u|^2)u = 0 \quad 0 < x < L$$

$$u(x) = e^{-ikx} + R e^{ikx} \quad x > L$$

$$u(x) = T e^{-ikx} \quad x < 0$$

The field amplitude has been scaled by the amplitude of the incident wave so that the nonlinear coefficient  $w$  is proportional to the input intensity. The transmission and reflection coefficients are respectively  $T$  and  $R$ . The randomness is modeled by  $\mu(x)$  which is a bounded, stationary, zero mean stochastic process. The parameter  $\epsilon$  allows us to control the size of the noise. Requiring continuity of  $u$  and  $u_x$  at  $x=0$  and  $x=L$  gives us a boundary value problem. We are considering the case where the zero-intensity homogeneous medium is matched to free space (index of refraction equal to one) to simplify the analysis.

A significant difficulty in the analysis is the possibility of bistability or lack of uniqueness. For a given value of  $w$  there may be several solutions to the time harmonic problem. Because of the non-uniqueness it is desirable to formulate the problem in a slightly different way. We convert it into a family of initial value problems with a parameter proportional to the transmitted intensity. This allows us to use methods from the asymptotic theory of stochastic equations when suitable scalings are introduced. When there is no randomness we find that bistability occurs when  $w \approx L^{-1/3}$  for  $L \rightarrow \infty$ . When random

inhomogeneities are present the analysis of the transmission problem is much more difficult and we first study the growth of the solutions of the family of initial value problems when the noise intensity  $\epsilon$  is small and  $L$  is large. In the linear case it is exponential but in the nonlinear case it is only algebraic. This has important implications for the onset of bistability when random inhomogeneities are present. It occurs for much smaller  $L$ , compared to the nonrandom case. We have made a full study of the statistical properties of the onset of bistability both analytically with asymptotic methods and numerically.

#### References

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## VIII. Faculty Resumes

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### **Home Address:**

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### **Education:**

Ph.D. (Mathematics), Courant Institute of Mathematical Sciences, NYU, June 1978.

M.S. (Mathematics), Courant Institute of Mathematical Sciences, NYU, February 1977.

B.S. (Mathematics), Michigan State University, June 1975.

### **Employment:**

1988-present Professor, Courant Institute of Mathematical Sciences, NYU

1984-1988 Associate Professor, Courant Institute of Mathematical Sciences, NYU

1983-1984 Assistant Professor, Courant Institute of Mathematical Sciences, NYU

1979-1982 Assistant Professor, Department of Mathematics, Stanford University

1978-1979 Visiting Member, Courant Institute of Mathematical Sciences, NYU

### **Editorial Board Member:**

Communications on Pure and Applied Mathematics

Mathematical Modeling and Numerical Analysis

SIAM Journal of Applied Mathematics

### **Awards:**

Alfred P. Sloan Research Fellow (1984-1989)

Hertz Foundation Graduate Fellow (1975-1978)

**Date and Place of Birth:**

April 29, 1954; Charleston, West Virginia.

**Professional Activities:**

Principal Investigator, URI Center for Analysis of Heterogeneous and Nonlinear Media (1986-present)

NSF Postdoctoral Fellowship Selection Committee (1987-1989)

Defense Science Study Group, Institute for Defense Analyses (1985-1988)

Organizer for Workshop on Mathematical Aspects of Vortex Dynamics, Leesburg, VA (1988)

Program Committee for Colloquium on Kinetic Theory of Gases, Paris (1988)

Program Committee for National SIAM Meeting, Boston (1986)

**Recent Invited Lectures:**

1. Workshop on Hydro. Behavior and Int. Particle Syst., IMA, Minneapolis, March, 1986
2. Workshop on Multiphase Flow, Leesburg, VA, June, 1986
3. Summerschool on Transport Th., Udine, Italy, June, 1986
4. AMS, San Antonio, February, 1987
5. Workshop on Vortex Methods, UCLA, June, 1987
6. IUTAM Sym. Vortex Motion, Tokyo, September, 1987
7. Singularities in PDEs, Tucson, March, 1988
8. Intl. Symp. Rarefied Gas Dynamics, Pasadena, June 1988
9. Intl. Workshop Math. Aspects Fluid and Plasma Dyn., Salice Terme, Italy, September, 1988

**Graeme Walter Milton**

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**Education:** Ph.D., Cornell University, 1985. Thesis title:

"Some Exotic Models in Statistical Physics."

Adviser: Professor M.E. Fisher.

M.Sc., Sydney University, 1982. Thesis title: "Bounds

on the Macroscopic Properties of Composite Materials."

Adviser: Dr. G.H. Derrick and Dr. R.C. McPhedran.

B.Sc., Sydney University, 1980 (in Physics with First

Class Honors)

**Academic and**

**Professional Honors:**

1988	David & Lucille Packard Fellowship for Science and Engineering
1988	Sloan Research Fellowship
1984-86	California Institute of Technology, Weingart Research Fellowship
1984	Chevron Oil Field Research Company unrestricted research grant
1981-83	Sydney University Traveling Scholarship for Graduate Study

- 1980 Commonwealth of Australia Postgraduate Scholarship
- 1980 Sydney University Medal in Physics
- 1980 Herbert Johnson Travel Grant
- 1979 Australian Institute of Physics, N.S.W. Branch Prize in Physics
- 1978 Deas-Thompson Scholarship for Physics, University of Sydney

**Professional Positions:**

- 1989-present Associate Professor, Courant Institute of Mathematical Sciences
- 1987-1989 Assistant Professor, Courant Institute of Mathematical Sciences
- 1984-1986 Weingart Postdoctoral Fellow, California Institute of Technology
- August, 1985 Consultant, Chevron Research Laboratories
- Spring, 1981 Teaching Assistant (part-time), Cornell University
- Summer, 1979 Research Assistant in Theoretical Chemistry, Sydney University
- 1978-1979 Teaching Assistant (part-time), Sydney University

**Recent Invited Lectures at Conferences:**

1. "The effective medium approximation is realizable as a rigorous limit", International Symposium on the Multiple Scattering of Waves in Random Media and Random Rough Surfaces, The Pennsylvania State University, University State Park, Pennsylvania, U.S.A., July 1985.
2. "Construction of continued fractions and bounds for effective tensors", Workshop on cross-disciplinary research in multiphase flow, Leesburg, Virginia, June 1986.
3. "Variational bounds on the effective moduli of anisotropic composites", Symposium on non-classical continuum mechanics: abstract techniques and applications, University of Durham, Durham, England, July 1986.
4. "A new field equation recursion method for bounding effective tensors of composites", Special session on non-linear partial differential equations at the American Mathematical Society annual meeting, San Antonio, Texas, January 1987.

**George C. Papanicolaou**

**Birth Date :** January 23, 1943.

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**Education:**

Ph.D. 1969, M.S 1967, Mathematics, New York University, Courant Institute

B.E.E 1965, Union College Schenectady N.Y.

**Employment:**

New York University, Courant Institute

Assistant Professor, 1969-1973

Associate Professor, 1973-1976

Professor, 1976-present

Director, Division of Wave Propagation and Applied Mathematics, Courant Institute 1979-present.

Visiting Member, IRIA, Rocquencourt, France, 12/74-2/75

Observatoire de Nice France, 3/75.

Visiting Member, IRIA, Rocquencourt, France 9/76-1/77.

Visiting Associate Professor, Cornell University (Spring term), 1976-1977

Visiting Scientist Exxon Research Corp. 10/83-1/84.

Visiting Professor, Univ. of Paris Dauphine 3/84

**Honors & Awards:**

Alfred P. Sloan Fellow, 1974-1976

John Simon Guggenheim Fellow 1983-1984

Invited speaker, International Congress of Mathematicians, 1986

**Recent invited lectures by George Papanicolaou**

Jan. 21-24, 1986, Conference on Nonlinear Phenomena, Los Alamos, invited hour speaker.

Feb. 11-12, 1986, RPI, Class of '27 special invited speaker.

March 17-24, 1986, IMA Univ. of Minnesota, Workshop on the hydrodynamic behavior of interacting particle systems, organizer and speaker.

May 19-23, 1986, Lisbon, Portugal, NATO Conference on Differential Equations, invited hour speaker.

June 2-4, 1986, Leesburg, Virginia, Conference on Multiphase Flow, organizer and speaker.

Aug. 4-12, 1986, International Congress of Mathematicians, Berkeley, California, invited speaker.

Aug. 18-Sept. 5, 1986, 12 lectures on waves in random media at the Ecole d'ete de St. Flour, France.

Dec. 1-5, 1986, IMA Univ. of Minn., workshop on math. methods in exploration geophysics, invited speaker.

Jan. 21-23, 1987, AMS annual meeting, special session on homogenization and oscillations, San Antonio, Texas, invited speaker.

Feb. 18-21, 1987, University of Arizona conference on random media, invited speaker.

March 4-5, 1987, two invited lectures at Purdue University.

March 16-20, 1987, RPI conference on multiphase flow, invited participant.

April 1-4, 1987, U. of Cambridge, England, three lectures at the annual meeting of British Probabilists.

May 4, 1987, invited lecture at the B. Levich Memorial Conference at the City University of New York.

Aug. 24-28, 1987, Equadiff, Xanthi, Greece, co-organizer and speaker.

Aug 24-28, 1987, special invited speaker at the European Statisticians Conference, Thessaloniki, Greece.

Sept. 14-16, 1987, Duke University and University of North Carolina at Chapel Hill, visit and lecture.

Nov. 14-15, 1987, invited speaker at the annual meeting of the Soc. for Natural Philosophy at Yale U.

March 14-18, 1988, Tuscon Arizona, meeting on singularities in PDE

April 15,16, 1988, Brown University, meeting in honor of Wendell Fleming

May 4-6, 1988, Los Alamos, Meeting on Nonlinearity and Localization

May 18-26, 1988, L'Aquila Italy, Meeting on Interacting particle systems

June 6-10, 1988, Paris France, Meeting in honor of J. L. Lions

July 18-29, 1988, Silivri Turkey, Workshop on stochastic analysis.

Oct 22-25, 1988, Cornell University, Workshop on multiphase flow.

Dec 6-9, 1988, Workshop on composite and random media, Leesburg Va.

Dec 14-16, 1988, Conference on computational methods in wave propagation, Nice, France.

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**Education:**

Chiao Tung University, BSc, 1946

MIT, MSc 1948

Harvard University, MSc, 1949

New York University, DSc, 1951

**Employment:**

1951-52, Research Associate, New York University

1952-55, Special Design Engineer, Foster Wheeler Corp.

1955-57, Research Associate

1957-58, Research Assistant Professor

1958-60, Research Associate Professor

1960-64, Research Professor Polytechnic Institute of Brooklyn

1964-68, Professor of Aeronautics and Astronautics

1968-Present, New York University

**Memberships:**

Sigma Xi

Sigma Gamma Tau

Society for Industrial and Applied Mathematics

American Physical Society

American Institute of Aeronautics and Astronautics

New York Academy of Sciences

**Recent Invited Talks of Lu Ting**

1. Minisymposium/28 " Analytical and Numerical Solutions of a Vortical Flow", Chairperson and Organizer L. Ting, in the First Intern. Conf. on Industrial and Appl. Math., Paris, July 1, 1987.

2. L. Ting, "Appropriate Boundary Conditions for Problems in Wave Propagation and in Fluid Dynamics" invited seminar, May 15, 1987 ,Air Force Wright Aeronautical Laboratory, Dayton, Ohio.

3. L. Ting, " Dynamics of Three Vortices, Revisited " Applied Mathematics Seminar, Stanford University, Dec. 13, 1987.

4. L. Ting, 10 Lectures series "Motion and Interaction of Vortices with Diffusive Core Structures" in commemoration of 75th year of the Aerodynamisches Institut, RWTH Aachen, April 25-29, 1988.

5. M. J. Miksis and L. Ting " Effects of Bubbly Layers on Wave Propagation", A Workshop on Theoretical Aspects of Multiphase Flow Phenomena, Cornell University Mathematical Sciences Institute, Oct. 22-25, 1988.

6. L. Ting and M. J. Miksis, " On Vortical Flow and Sound Generation", EUROMECH Colloquium 247, Vortex Interaction Noise, Goettingen, June 28 - July 1, 1989.

7. L. Y. Cheng and L. Ting, " Void Distribution in a Shallow Bubbling Pool", 26th National Heat Transfer Conference, Philadelphia, August 6-9, 1989.

## IX. Publications of the Center

### James G. Berryman

- [1] James G. Berryman, Graeme W. Milton - "Microgeometry of Random Composites and Porous Media" J. Phys. D: Appl. Phys. 21 (1988) 87-94.
- [2] James G. Berryman, Lawrence Livermore National Laboratory - "Weighted Least-Squares Criteria for Seismic Traveltime Tomography" - preprint Geophys. J. Roy. Astron. Soc.

### Russel E. Caflisch

- [1] Lectures on the Mathematical Theory of Multi-Phase Flow (with J. Rubinstein). CIMS Lecture Note Series (1986).
- [2] "Equilibrium for Radiation in a Homogeneous Plasma" (with C.D. Levermore), Phys. Fluids, 29 (1986) 748-752.
- [3] "Mathematical Method for the Equations of Kinetics of Gases," (with C.V.M. Van der Mee) Transport Th. Stat. Physics 15 (1986) 353-367.
- [4] "Stability of Shock Wave Profiles for a Model Boltzmann," Equation (with T.P. Liu) 15th RGD Symposium, Grado, (1986) 3-9.
- [5] "Statistical Properties of Sedimenting Suspensions and the Point Particle Approximation," (with J.H.C. Luke) in *Advances in Multiphase Flow and Related Problems*, ed. Papanicolaou, SIAM (1986) 23-34.
- [6] "The Milne and Kramers Problems for the Boltzmann Equation of a Hard Sphere Gas" (with C. Bardos, B. Nicolaenko), Comm. Pure Appl. Math., 39 (1986) 323-352.
- [7] "Long Time Existence for Slightly Perturbed Vortex Sheets" (with O. Orellana), Comm. Pure Appl. Math., 39 (1986) 807-838.
- [8] "Asymptotic Expansions of Solutions for the Boltzmann Equation," J. Transport Th. Stat. Phys., 16 (1987) 701-725.
- [9] "Statistical Modeling of a Dilute Fluid-Particle Suspension," in *Hydrodynamic Behavior and Interacting Particle Systems*, IMA vol. 9, ed. Papanicolaou, Springer (1987) 1-12.

- [10] "Asymptotics of the Boltzmann Equation and Fluid Dynamics," in *Kinetic Theory and Gas Dynamics* CISM No. 239 (1988) ed. C. Cercignani, pp. 95-133.
- [11] "Long Time Existence and Singularity Formation for Vortex Sheets," Workshop on Vortex Methods, ed. Anderson and Greengard.
- [12] "Nonlinear Analysis for the Evolution of Vortex Sheets," in *Vortex Motion*, Proc. IUTAM Symposium on Fundamental Aspects of Vortex Motion, ed. H. Hasimoto and T. Kambe, North-Holland (1988) 75-77 and in *Fluid Dyn. Research* 3 (1988) 75-77.
- [13] "Well-Posedness of Initial and Boundary Value Problems for the Boltzmann Equation," Proc. 16th Intern. Symp. Rarefied Gas Dyn. (1988)
- [14] "Singularity Formation for Vortex Sheets and Hyperbolic Equations," 3rd Intern. Wrkshp. on Math. Aspects Fluids and Plasmas (1988)
- [15] "Mathematical Analysis for Vortex Dynamics," Proc. Workshop on Math. Aspects of Vortex Dyn. (1988) SIAM.
- [16] "Stability of Shock Waves for the Broadwell Equations" (with T.-P. Liu) *Comm. Math. Phys.* 114 (1988) 103-130.
- [17] "Periodic Solutions for Three Sedimenting Spheres" (with J. Luke, C. Lim and A. Sangani) *Phys. Fl.* 31 (1988) 3175-3179.
- [18] "Singularity Formation and Ill-Posedness for Vortex Sheets" (with O. Orellana) *SIAM J. Math. Anal.* to appear
- [19] "Convergence of the Vortex Method for Vortex Sheets" (with J. Lowengrub) *SIAM J. Num. Anal.* to appear.
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